ALGEBRAIC STRUCTURES IN GROUP-THEORETICAL FUSION CATEGORIES

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Abstract. It was shown by Ostrik (2003) and Natale (2017) that a collection of twisted group algebras in a pointed fusion category serve as explicit Morita equivalence class representatives of indecomposable, separable algebras in such categories. We generalize this result by constructing explicit Morita equivalence class representatives of indecomposable, separable algebras in group-theoretical fusion categories. This is achieved by providing the 'free functor' Φ from a pointed fusion category to a group-theoretical fusion category with a monoidal structure. Our algebras of interest are then constructed as the image of twisted group algebras under Φ. We also show that twisted group algebras admit the structure of Frobenius algebras in a pointed fusion category, and we establish a Frobenius monoidal structure on Φ as well. As a consequence, our algebras are Frobenius algebras in a group-theoretical fusion category, and like twisted group algebras in the pointed case, they also enjoy several good algebraic properties.

Contents

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1. Introduction

The goal of this work is to construct explicit algebras that represent Morita equivalence classes in group-theoretical fusion categories, and that possess good algebraic properties. Throughout, we assume that $\Bbbk$ is an algebraically closed field of characteristic 0, and an unadorned $\otimes$ denotes $\otimes_{\Bbbk}$.

A group-theoretical fusion category is a certain kind of monoidal category whose construction depends on group-theoretic data, and we will restrict our attention to such categories below. But for now let us discuss the prevalence of Morita equivalence of algebras in monoidal categories in general. Recall that two rings are said to be *Morita equivalent* if their categories of modules are equivalent as abelian categories. Many nice properties are preserved under such an equivalence including the Noetherian, (semi)simple, (semi)hereditary, and (semi)prime conditions [24, Chapter 7]. The notion of Morita equivalence has been upgraded for algebras of various types, and is used in several areas including C^*-algebras [3], Poisson geometry [34], and various subfields of physics [7, 15, 31]; we will discuss [15] below. In all of these cases, one is studying the Morita equivalence of algebras (or, of algebra objects) in a fixed monoidal category. We provide a review of this categorical terminology, including the definition of Morita equivalence and of special types of algebras under consideration, in Section 2.

A special use of Morita equivalence occurs in two-dimensional *rational conformal field theories* (rCFTs). These are certain quantum field theories that display conformal symmetry, and they have inspired vital mathematical structures such as vertex operator algebras [2] and modular tensor categories (MTCs) [1, Chapter 3]. Often, algebras in MTCs provide a useful way of classifying and describing the physical quantities in rCFTs. In particular, *full rCFTs* [15] are completely fixed by pairs (ν, A), where ν is a rational vertex operator algebra encoding the symmetries of the rCFT, and A is a separable, symmetric Frobenius algebra in $\mathrm{Rep}\,(\nu)$ (which is an MTC [21]). This algebra A is the algebra of boundary fields associated to one given boundary condition of the full rCFT. Moreover, the algebras arising from boundary conditions of the full rCFT are all Morita equivalent. So, essentially, the collection of full rCFTs are in bijection with Morita equivalence classes of separable, symmetric Frobenius algebras in MTCs of the form $\mathrm{Rep}\,(\nu)$.

Now returning to the goal of our work, we discuss how the aim is resolved partially for an arbitrary fusion category $\mathcal{C}$ by work of V. Ostrik [30]. Two algebras in $\mathcal{C}$ are said to be *Morita equivalent* if their categories of (right) modules in $\mathcal{C}$ are equivalent as (left) $\mathcal{C}$-module categories; see Section 2.3. The main result of [30] states that any $\mathcal{C}$-module category $\mathcal{M}$ is equivalent to the category of modules over some algebra A in $\mathcal{C}$, and the algebra A used in the proof of this result is an *internal End* of any nonzero object of $\mathcal{M}$ (see [30, Section 3.2]). It is also shown that this internal End A can be taken to be connected [Definition 2.10], but no other good algebraic properties of A are established nor is the construction of A explicit. In contrast, we restrict our attention to certain types of fusion categories that depend on group-theoretic data and, using a construction different than internal Ends, we produce Morita equivalence representatives of algebras in these categories that depend explicitly on this group-theoretic data.

Such a strategy was used to resolve the goal in the setting of pointed fusion categories, that is, for the categories Vec_G^ω, with G a finite group and $\omega \in H^3(G, \Bbbk^\times)$, consisting of G-graded $\Bbbk$-vector spaces with associativity constraint ω. The simple objects of Vec_G^ω are 1-dimensional $\Bbbk$-vector spaces, denoted by $\{\delta_g\}_{g \in G}$, with G-grading $(\delta_g)_x = \delta_{g,x}\ \Bbbk$, for $g, x \in G$. Indeed, we have the following construction and result due to work of V. Ostrik and work of S. Natale.

Definition 1.1. Let L be a subgroup of G so that $\omega|_{L^{\times 3}}$ is trivial, and take a 2-cochain $\psi \in C^2(L, \Bbbk^\times)$ so that $d\psi = \omega|_{L^{\times 3}}$. The *twisted group algebra* $A(L,\psi)$ in Vec_G^ω is $\bigoplus_{g \in L} \delta_g$, with multiplication given by $\delta_g \otimes \delta_{g'} \mapsto \psi(g,g')\delta_{gg'}$.

Theorem 1.2. [29, Example 2.1] [8, Example 9.7.2] [28] *A collection of twisted group algebras $A(L,\psi)$ serve as Morita equivalence class representatives of indecomposable, separable algebras in the pointed fusion category Vec_G^ω.* □

The first of our results is that we establish a Frobenius algebra structure on the twisted group algebras and study related algebraic properties. See Definition 2.10 for a description of the properties mentioned below.

Proposition 1.3 (Propositions 5.7 and 5.9)**.** *The twisted group algebras $A(L,\psi)$ admit the structure of a Frobenius algebra in Vec_G^ω. They are also indecomposable and separable in Vec_G^ω, are connected, are special, and are symmetric if and only if $\omega(g^{-1}, g, g^{-1}) = 1$ for each $g \in L$.* □

Now we turn our attention to *group-theoretical fusion categories*. Introduced by P. Etingof, D. Nikshych, and V. Ostrik in [10, Section 8.8], these are the categories $\mathcal{C}(G,\omega,K,\beta)$ consisting of $A(K,\beta)$-bimodules in Vec_G^ω, for G and ω as above, and with K a subgroup of G so that $\omega|_{K^{\times 3}}$ is trivial, and $\beta \in C^2(K, \Bbbk^\times)$ so that $d\beta = \omega|_{K^{\times 3}}$. (See also [8, Section 9.7].) Group-theoretical fusion categories are a vital part of the classification program of general fusion categories (see, e.g., [11, Theorem 9.2] and [8, Section 9.13]), and due to their explicit construction, they also serve as a go-to testing ground for results about fusion categories (see, e.g., [9, Section 5], [13, Corollary 4.4], [18], [19, Section 4], [27], [29]).

Towards our goal of constructing nice Morita equivalence class representatives of algebras in group-theoretical fusion categories, we start in a more general setting and consider the 'free' functor from a monoidal category $\mathcal{C}$ to a category of bimodules in $\mathcal{C}$, and impose on this functor further structure (see Definition 2.2).

Theorem 1.4 (Theorem 3.2)**.** *Let $(\mathcal{C}, \otimes)$ be a monoidal category, and let A be a special Frobenius algebra in $\mathcal{C}$. Let ${}_A\mathcal{C}_A$ denote the monoidal category of A-bimodules in $\mathcal{C}$. Then, the functor $\Phi : \mathcal{C} \to {}_A\mathcal{C}_A$, which sends objects X to $(A \otimes X) \otimes A$, and morphisms φ to $(\mathrm{id}_A \otimes \varphi) \otimes \mathrm{id}_A$, admits the structure of a Frobenius monoidal functor.* □

The result above enables us to define algebraic structures that will fulfill our goal.

Definition-Theorem 1.5 (Definition 6.3, Theorem 6.4)**.** *Using the free functor* Φ *above in the case when* $\mathcal{C} = \mathsf{Vec}_G^\omega$ *and* $A = A(K,\beta)$*, we define the* twisted Hecke algebra $A^{K,\beta}(L,\psi)$ *to be the algebra* $\Phi(A(L,\psi))$ *in* $\mathcal{C}(G,\omega,K,\beta)$*. It admits the structure of a Frobenius algebra in* $\mathcal{C}(G,\omega,K,\beta)$*, an explicit description of which is known.* □

The terminology is due to the fact that simple objects of group-theoretical fusion categories $\mathcal{C}(G,\omega,K,\beta)$ are in part parameterized by K-double cosets in G (see Lemma 6.2), and the multiplication of $A^{K,\beta}(L,\psi)$ is twisted by cocycles β and ψ. Twisted Hecke algebras also enjoy several nice algebraic properties.

Proposition 1.6 (Proposition 6.9)**.** *The twisted Hecke algebras* $A^{K,\beta}(L,\psi)$ *are indecomposable and separable (Frobenius algebras) in* $\mathcal{C}(G,\omega,K,\beta)$*, and are also special.* □

We provide a precise condition describing when twisted Hecke algebras are connected in Proposition 6.11; in general, the connected property does not hold. In any case, the twisted Hecke algebras in $\mathcal{C}(G,\omega,K,\beta)$ are nearly as nice algebraically as the twisted group algebras in Vec_G^ω; cf. Proposition 1.3. We inquire about the symmetric property for twisted Hecke algebras in Question 6.10, which involves understanding the explicit rigidity structure of group-theoretical fusion categories (see Question 2.22); this is reserved for future work.

Finally, our goal is achieved as follows.

Theorem 1.7 (Theorem 7.4)**.** *A collection of twisted Hecke algebras* $A^{K,\beta}(L,\psi)$ *serve as Morita equivalence class representatives of indecomposable, separable algebras in the group-theoretical fusion category* $\mathcal{C}(G,\omega,K,\beta)$*.* □

An application of this result to P. Etingof, R. Kinser, and the last author's study of tensor algebras in group-theoretical fusion categories [9] is discussed in Remark 7.6 and Example 7.7.

Theorem 1.7 is achieved by introducing the notion of a *Morita preserving monoidal functor* [Theorem 4.1, Definition 4.3] and by establishing the following general result.

Theorem 1.8 (Theorem 4.9)**.** *Let* $\mathcal{C}$ *be a rigid monoidal category. Take a special Frobenius algebra* A *in* $\mathcal{C}$*, and take algebras* B*,* B' *in* $\mathcal{C}$*. Recall the monoidal functor* Φ *from Theorem 1.4. Then,* B *and* B' *are Morita equivalent as algebras in* $\mathcal{C}$ *if and only if* $\Phi(B)$ *and* $\Phi(B')$ *are Morita equivalent as algebras in* ${}_A\mathcal{C}_A$*.* □

Indeed, with Theorem 1.2 (due to Ostrik and Natale) and Proposition 1.3, Theorem 1.8 provides the crucial step in proving Theorem 1.7 by setting $\mathcal{C} = \mathsf{Vec}_G^\omega$, $A = A(K,\beta)$, $B = A(L,\psi)$, $B' = A(L',\psi')$.

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2. Preliminaries on monoidal categories

In this section, we provide background information and preliminary results on monoidal categories and monoidal functors in Section 2.1, on algebraic structures in monoidal categories in Section 2.2, and on categories of modules and bimodules over algebras in monoidal categories in Sections 2.3 and 2.4. In Section 2.5, we establish preliminary results on Morita equivalence of algebras in monoidal categories.

To begin, take an abelian category $\mathcal{C}$ with a zero object 0 throughout this article. By $X \in \mathcal{C}$ we mean that X is an object of $\mathcal{C}$. A nonzero $X \in \mathcal{C}$ is *simple* if 0 and X are its only subobjects. A category $\mathcal{C}$ is *semisimple* if every object is a direct sum of simple objects. We say that $X \in \mathcal{C}$ is *indecomposable* if it is nonzero and cannot be decomposed as the direct sum of nonzero subobjects. Simple objects in $\mathcal{C}$ are indecomposable; the converse holds when $\mathcal{C}$ is semisimple. We assume that all categories in this work are *locally small*, i.e., for any objects $X, Y \in \mathcal{C}$ the collection of morphisms from X to Y is a set.

Moreover, for a field $\Bbbk$, a $\Bbbk$-linear category $\mathcal{C}$ is *locally finite* if each Hom space is a finite-dimensional $\Bbbk$-vector space and if every object has finite length. We also say that a $\Bbbk$-linear category $\mathcal{C}$ is *finite* if it is equivalent to the category of finite-dimensional modules over some finite-dimensional $\Bbbk$-algebra.

2.1. **Monoidal categories and functors.**

Definition 2.1. (see, e.g., [8, Definition 2.2.8]) A *monoidal category* $\mathcal{C}$ consists of the following data:

- a category $\mathcal{C}$,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- an object $\mathbb{1} \in \mathcal{C}$,
- a natural isomorphism $\alpha_{X,X',X''} : (X \otimes X') \otimes X'' \xrightarrow{\sim} X \otimes (X' \otimes X'')$ for each $X, X', X'' \in \mathcal{C}$,

- natural isomorphisms $l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$, $r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$ for each $X \in \mathcal{C}$,

such that the pentagon and triangle axioms are satisfied [8, (2.2),(2.10)].

Definition 2.2. [32, page 85] [6] [33, (6.46), (6.47)]
Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{*,*,*}, l_*, r_*)$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha_{*,*,*}, l_*, r_*)$ be monoidal categories.

(a) A *monoidal functor* $(F, F_{*,*}, F_0) : \mathcal{C} \to \mathcal{D}$ consists of the following data:
  - a functor $F : \mathcal{C} \to \mathcal{D}$,
  - a natural transformation $F_{X,X'} : F(X) \otimes_{\mathcal{D}} F(X') \to F(X \otimes_{\mathcal{C}} X')$ for all $X, X' \in \mathcal{C}$,
  - a morphism $F_0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$,

  that satisfy the following associativity and unitality constraints,
$$F_{X,X'\otimes_{\mathcal{C}}X''}\,(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F_{X',X''})\,\alpha_{F(X),F(X'),F(X'')} = F(\alpha_{X,X',X''})\,F_{X\otimes_{\mathcal{C}}X',X''}\,(F_{X,X'} \otimes_{\mathcal{D}} \mathrm{id}_{F(X'')}),$$
$$\begin{aligned} F(l_X)^{-1}\,l_{F(X)} &= F_{\mathbb{1}_{\mathcal{C}},X}\,(F_0 \otimes_{\mathcal{D}} \mathrm{id}_{F(X)}),\\ F(r_X)^{-1}\,r_{F(X)} &= F_{X,\mathbb{1}_{\mathcal{C}}}\,(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F_0).\end{aligned}$$

(b) A *comonoidal functor* $(F, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ consists of the following data:
  - a functor $F : \mathcal{C} \to \mathcal{D}$,
  - a natural transformation $F^{X,X'} : F(X \otimes_{\mathcal{C}} X') \to F(X) \otimes_{\mathcal{D}} F(X')$ for all $X, X' \in \mathcal{C}$,
  - a morphism $F^0 : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ in $\mathcal{D}$,

  that satisfy the following coassociativity and counitality constraints,
$$\alpha^{-1}_{F(X),F(X'),F(X'')}\,(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F^{X',X''})\,F^{X,X'\otimes_{\mathcal{C}}X''} = (F^{X,X'} \otimes_{\mathcal{D}} \mathrm{id}_{F(X'')})\,F^{X\otimes_{\mathcal{C}}X',X''}\,F(\alpha^{-1}_{X,X',X''}),$$
$$\begin{aligned} F(l_X) &= l_{F(X)}\,(F^0 \otimes_{\mathcal{D}} \mathrm{id}_{F(X)})\,F^{\mathbb{1}_{\mathcal{C}},X},\\ F(r_X) &= r_{F(X)}\,(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F^0)\,F^{X,\mathbb{1}_{\mathcal{C}}}.\end{aligned}$$

(c) A *Frobenius monoidal functor* $(F, F_{*,*}, F_0, F^{*,*}, F^0)$ from $\mathcal{C}$ to $\mathcal{D}$ is a functor where $(F, F_{*,*}, F_0)$ is monoidal and $(F, F^{*,*}, F^0)$ is comonoidal, such that for all $X, X', X'' \in \mathcal{C}$:
$$(F_{X,X'} \otimes_{\mathcal{D}} \mathrm{id}_{F(X'')})\alpha^{-1}_{F(X),F(X'),F(X'')}(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F^{X',X''}) = F^{X\otimes_{\mathcal{C}}X',X''}F(\alpha^{-1}_{X,X',X''})F_{X,X'\otimes_{\mathcal{C}}X''},$$
$$(\mathrm{id}_{F(X)} \otimes_{\mathcal{D}} F_{X',X''})\alpha_{F(X),F(X'),F(X'')}(F^{X,X'} \otimes_{\mathcal{D}} \mathrm{id}_{F(X'')}) = F^{X,X'\otimes_{\mathcal{C}}X''}F(\alpha_{X,X',X''})F_{X\otimes_{\mathcal{C}}X',X''}.$$

Here, 'monoidal' means 'lax monoidal' in other references. Strong monoidal functors are monoidal functors where $F_{*,*}$ are F_0 are isomorphisms in $\mathcal{D}$, and we do not require this condition here.

Definition 2.3. (see, e.g., [8, Sections 7.1, 7.2]) Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{*,*,*}, l_*, r_*)$ be a monoidal category.

(a) A *left $\mathcal{C}$-module category* is a category $\mathcal{M}$ equipped with
  - a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$,

- natural isomorphisms for associativity

$$m_{X,Y,M} : (X \otimes Y) \otimes M \xrightarrow{\sim} X \otimes (Y \otimes M), \quad \forall X, Y \in \mathcal{C},\ M \in \mathcal{M}$$

  satisfying the pentagon axiom, and
- for each $M \in \mathcal{M}$ a natural isomorphism $\mathbf{1} \otimes M \xrightarrow{\sim} M$ satisfying the triangle axiom.

Right $\mathcal{C}$-module categories are defined analogously.

(b) A module category $\mathcal{M}$ over $\mathcal{C}$ is *indecomposable* if it is nonzero and is not equivalent to a direct sum of two nontrivial module categories over $\mathcal{C}$.

(c) Let $\mathcal{M}$ and $\mathcal{N}$ be two left $\mathcal{C}$-module categories. A *(left) $\mathcal{C}$-module functor* from $\mathcal{M}$ to $\mathcal{N}$ is a functor $F : \mathcal{M} \to \mathcal{N}$ equipped with a natural isomorphism $s_{X,M} : F(X \otimes M) \xrightarrow{\sim} X \otimes F(M)$ for each $X \in \mathcal{C}$, $M \in \mathcal{M}$ satisfying the pentagon and triangle axioms. *Right $\mathcal{C}$-module functors* are defined analogously.

(d) An *equivalence of $\mathcal{C}$-module categories* is a $\mathcal{C}$-module functor (F, s) so that $F : \mathcal{M} \to \mathcal{N}$ is an equivalence of categories.

Now we recall terminology for monoidal categories with dual objects.

Definition 2.4. Let $\mathcal{C}$ be a monoidal category. An object in $\mathcal{C}$ is called *rigid* if it has left and right duals. Namely, for each $X \in \mathcal{C}$, there exists objects X^* and ${}^*X \in \mathcal{C}$ so that we have co/evaluation maps

$$\begin{aligned} \mathsf{ev}_X &: X^* \otimes X \to \mathbb{1}, & \mathsf{coev}_X &: \mathbb{1} \to X \otimes X^*, \\ \mathsf{ev}'_X &: X \otimes {}^*X \to \mathbb{1}, & \mathsf{coev}'_X &: \mathbb{1} \to {}^*X \otimes X, \end{aligned}$$

satisfying compatibility conditions [8, (2.43)–(2.46)]. The monoidal category $\mathcal{C}$ is called *rigid* if each of its objects is rigid.

Later in Sections 5, 6, and 7, we will focus our attention on $\Bbbk$-linear categories. So, consider the following terminology.

Definition 2.5. [8, Sections 2.1, 2.10 and 4.1] Let $\mathcal{C}$ be a $\Bbbk$-linear, locally-finite, monoidal category. (Recall we assume that $\mathcal{C}$ is abelian.)

(a) We call $\mathcal{C}$ a *multi-tensor category* if it is also rigid. If, in addition, $\mathsf{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$ (i.e., if $\mathbb{1}$ is a simple object of $\mathcal{C}$), then $\mathcal{C}$ is a *tensor category.*

(b) A *multi-fusion category* is a finite semisimple multi-tensor category. A *fusion category* is a multi-fusion category with $\mathsf{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$, i.e., a finite semisimple tensor category.

With extra structure on $\mathcal{C}$, we require more structure of its module categories. The notion below will be of use later.

Definition 2.6. [30, Section 2.2, Definition 2.6] A *module category over a fusion category* $\mathcal{C}$ is a $\mathcal{C}$-module category $\mathcal{M}$ as in Definition 2.3 that is, in addition, semisimple, $\Bbbk$-linear, and abelian so that its bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ is bilinear on morphisms and is exact.

2.2. **Algebraic structures in monoidal categories.** Now we recall the notion of an algebra, a coalgebra, and a Frobenius algebra in a monoidal category. For general information, see [16, Section 2], [30, Section 3], [8, Section 7.8], and references within.

Definition 2.7 ($\mathsf{Alg}(\mathcal{C})$, $\mathsf{Coalg}(\mathcal{C})$, $\mathsf{FrobAlg}(\mathcal{C})$)**.** Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ be a monoidal category.

(a) An *algebra* in $\mathcal{C}$ is a triple (A, m, u), with $A \in \mathcal{C}$, and $m : A \otimes A \to A$ (multiplication), $u : \mathbb{1} \to A$ (unit) being morphisms in $\mathcal{C}$, satisfying unitality and associativity constraints:

$$m(m \otimes \mathrm{id}) = m(\mathrm{id} \otimes m)\alpha_{A,A,A}, \qquad m(u \otimes \mathrm{id}) = l_A, \qquad m(\mathrm{id} \otimes u) = r_A.$$

A *morphism* of algebras (A, m_A, u_A) to (B, m_B, u_B) is a map $f : A \to B$ in $\mathcal{C}$ so that $fm_A = m_B(f \otimes f)$ and $fu_A = u_B$. Algebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{Alg}(\mathcal{C})$.

(b) A *coalgebra* in $\mathcal{C}$ is a triple (C, Δ, ε), where $C \in \mathcal{C}$, and $\Delta : C \to C \otimes C$ (comultiplication) and $\varepsilon : C \to \mathbb{1}$ (counit) are morphisms in $\mathcal{C}$, satisfying counitality and coassociativity constraints:

$$\alpha_{C,C,C}(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \qquad (\varepsilon \otimes \mathrm{id})\Delta = l_C^{-1}, \qquad (\mathrm{id} \otimes \varepsilon)\Delta = r_C^{-1}.$$

A *morphism* of coalgebras $(C, \Delta_C, \varepsilon_C)$ to $(D, \Delta_D, \varepsilon_D)$ is a morphism $g : C \to D$ in $\mathcal{C}$ so that $\Delta_D g = (g \otimes g)\Delta_C$ and $\varepsilon_D g = \varepsilon_C$. Coalgebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{Coalg}(\mathcal{C})$.

(c) A *Frobenius algebra* in $\mathcal{C}$ is a tuple $(A, m, u, \Delta, \varepsilon)$, where $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$ and $(A, \Delta, \varepsilon) \in \mathsf{Coalg}(\mathcal{C})$, so that

$$(m \otimes \mathrm{id})\alpha^{-1}_{A,A,A}(\mathrm{id} \otimes \Delta) = \Delta m = (\mathrm{id} \otimes m)\alpha_{A,A,A}(\Delta \otimes \mathrm{id}).$$

A *morphism* of Frobenius algebras in $\mathcal{C}$ is a morphism in $\mathcal{C}$ that lies in both $\mathsf{Alg}(\mathcal{C})$ and $\mathsf{Coalg}(\mathcal{C})$. Frobenius algebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{FrobAlg}(\mathcal{C})$.

Remark 2.8. Alternatively, a Frobenius algebra in $\mathcal{C}$ is a tuple (A, m, u, p, q), where $p : A \otimes A \to \mathbb{1}$ and $q : \mathbb{1} \to A \otimes A$ are morphisms in $\mathcal{C}$ satisfying an invariance condition, $p(\mathrm{id}_A \otimes m)\alpha_{A,A,A} = p(m \otimes \mathrm{id}_A)$, and the Snake Equation,

$$r_A \; (\mathrm{id}_A \otimes p) \; \alpha_{A,A,A} \; (q \otimes \mathrm{id}_A) \; l_A^{-1} \;=\; \mathrm{id}_A \;=\; l_A \; (p \otimes \mathrm{id}_A) \; \alpha^{-1}_{A,A,A} \; (\mathrm{id}_A \otimes q) \; r_A^{-1}.$$

In this case, we call p a *non-degenerate pairing*. To convert from (A, m, u, p, q) to $(A, m, u, \Delta, \varepsilon)$ in Definition 2.7(c), take

$$\Delta := (m \otimes \mathrm{id}_A) \; \alpha^{-1}_{A,A,A} \; (\mathrm{id}_A \otimes q) \; r_A^{-1} \quad \text{and} \quad \varepsilon := p \; (u \otimes \mathrm{id}_A) \; r_A^{-1}.$$

On the other hand, to convert from $(A, m, u, \Delta, \varepsilon)$ to (A, m, u, p, q), take

$$p := \varepsilon_A m_A \quad \text{and} \quad q := \Delta_A u_A.$$

In fact, one can see that a Frobenius algebra in a monoidal category is a self-dual object with evaluation and coevaluation maps given by p and q, respectively. See [17] an [22, Section 2.3] for more details.

We have another equivalent definition of a Frobenius algebra in $\mathcal{C}$ in the case when $\mathcal{C}$ is rigid; this is given in Remark 2.16 below.

Next, we recall how the functors of Definition 2.2 preserve the algebraic structures in Definition 2.7.

Proposition 2.9. [32, p.100-101] [33, Lemma 2.1] [6, Corollary 5] [23, Prop. 2.13]

(a) *Let* $(F, F_{*,*}, F_0) : \mathcal{C} \to \mathcal{D}$ *be a monoidal functor. If* $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$*, then*
$$(F(A),\ F(m)F_{A,A},\ F(u)F_0) \in \mathsf{Alg}(\mathcal{D}).$$

(b) *Let* $(F, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ *be a comonoidal functor. If* $(C, \Delta, \varepsilon) \in \mathsf{Coalg}(\mathcal{C})$*, then*
$$(F(C),\ F^{C,C}F(\Delta),\ F^0F(\varepsilon)) \in \mathsf{Coalg}(\mathcal{D}).$$

(c) *Let* $(F, F_{*,*}, F_0, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ *be a Frobenius monoidal functor. If* $(A, m, u, \Delta, \varepsilon) \in \mathsf{FrobAlg}(\mathcal{C})$*, then*
$$(F(A),\ F(m)F_{A,A},\ F(u)F_0,\ F^{A,A}F(\Delta),\ F^0F(\varepsilon)) \in \mathsf{FrobAlg}(\mathcal{D}). \qquad \square$$

Some properties of the structures in Definition 2.7 of interest here are given below.

Definition 2.10. Take $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ a $\Bbbk$-linear, monoidal category.

(a) $A \in \mathsf{Alg}(\mathcal{C})$ is *indecomposable* if it is not isomorphic to a direct sum of non-trivial algebras in $\mathcal{C}$.

(b) $A \in \mathsf{Alg}(\mathcal{C})$ is *connected* (or *haploid*) if $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) = 1$, that is, if the unit of A is unique up to scalar multiple.

(c) $A \in \mathsf{Alg}(\mathcal{C})$ is *separable* if there exists a morphism $\Delta' : A \to A \otimes A$ in $\mathcal{C}$ so that $m\Delta' = \mathrm{id}_A$ as maps in $\mathcal{C}$ with
$$(\mathrm{id}_A \otimes m)\alpha_{A,A,A}(\Delta' \otimes \mathrm{id}_A) \ = \ \Delta' m \ = \ (m \otimes \mathrm{id}_A)\alpha^{-1}_{A,A,A}(\mathrm{id}_A \otimes \Delta').$$

(d) $(A, m, u, \Delta, \varepsilon) \in \mathsf{FrobAlg}(\mathcal{C})$ is *special* if $m\Delta = \mathrm{id}_A$ and $\varepsilon u = \varphi\ \mathrm{id}_{\mathbb{1}}$ for a nonzero $\varphi \in \Bbbk$.

Remark 2.11. (a) The displayed equations in Definition 2.10(c) above are the requirement that Δ' is both a left and a right A-module map, so that m splits as a map of A-bimodules in $\mathcal{C}$; see Sections 2.3 and 2.4.

(b) The special Frobenius condition above implies separability.

In order to ask for a Frobenius algebra to be symmetric in the categorical sense, we work in a rigid monoidal category; see [15, 17].

Definition 2.12. Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r, {}^*(), ()^*)$ be a rigid monoidal category. We say that a Frobenius algebra $(A, m, u, \Delta, \varepsilon)$ is *symmetric* if ${}^*A = A^*$ as objects in $\mathcal{C}$, and if εm is equal to
$$\Omega_A := \mathsf{ev}_A\ (\mathrm{id}_{A^*} \otimes l_A)\ (\mathrm{id}_{{}^*A = A^*} \otimes \varepsilon m \otimes \mathrm{id}_A)\ (\alpha_{{}^*A,A,A} \otimes \mathrm{id}_A)\ ((\mathsf{coev}'_A \otimes \mathrm{id}_A)\ l_A^{-1} \otimes \mathrm{id}_A)$$
as morphisms in $\mathcal{C}$.

Example 2.13. Suppose that $\mathcal{C} = \mathsf{Vec}_{\Bbbk}$, the category of finite-dimensional $\Bbbk$-vector spaces. Then the definition above recovers the usual notion of a symmetric Frobenius $\Bbbk$-algebra. Indeed, for A a Frobenius $\Bbbk$-algebra and $x, y \in A$ we get that

$$\begin{aligned}\Omega_A(x \otimes y) &= \mathsf{ev}_A(\mathrm{id}_{A^*} \otimes l_A)(\mathrm{id}_{{}^*A = A^*} \otimes \varepsilon m \otimes \mathrm{id}_A)(\textstyle\sum_i ({}^*e_i \otimes (e_i \otimes x)) \otimes y) \\ &= \mathsf{ev}_A(\mathrm{id}_{A^*} \otimes l_A)(\textstyle\sum_i (e_i^* \otimes \varepsilon m(e_i \otimes x)) \otimes y) \\ &= \mathsf{ev}_A(\textstyle\sum_i \varepsilon m(e_i \otimes x)(e_i^* \otimes y) \\ &= \varepsilon m(y \otimes x).\end{aligned}$$

2.3. **Categories of modules over algebras.** Fix $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ a monoidal category. Now we turn our attention to modules over algebras in $\mathcal{C}$. For more details, see [30, Section 3] and [8, Section 7.8].

Definition 2.14 (ρ_M, ρ_M^A, λ_M, λ_M^A, $\mathcal{C}_A$, ${}_A\mathcal{C}$)**.** Take $A := (A, m_A, u_A)$, an algebra in $\mathcal{C}$. A *right A-module in $\mathcal{C}$* is a pair (M, ρ_M), where $M \in \mathcal{C}$, and $\rho_M := \rho_M^A : M \otimes A \to M$ is a morphism in $\mathcal{C}$ so that

$$\rho_M(\rho_M \otimes \mathrm{id}_A) = \rho_M(\mathrm{id}_M \otimes m_A)\alpha_{M,A,A} \quad \text{and} \quad r_M = \rho_M(\mathrm{id}_M \otimes u_A).$$

A *morphism* of right A-modules in $\mathcal{C}$ is a morphism $f : M \to N$ in $\mathcal{C}$ so that $f\rho_M = \rho_N(f \otimes \mathrm{id}_A)$. Right A-modules in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathcal{C}_A$. The category ${}_A\mathcal{C}$ of *left A-modules* $(M, \lambda_M := \lambda_M^A : A \otimes M \to M)$ *in $\mathcal{C}$* is defined likewise.

We have that $\mathcal{C}_A$ is a left $\mathcal{C}$-module category: for $X \in \mathcal{C}$ and $(M, \rho_M) \in \mathcal{C}_A$, the bifunctor $\mathcal{C} \times \mathcal{C}_A \to \mathcal{C}_A$ is defined by

$$(X \otimes M) \otimes A \xrightarrow{\alpha_{X,M,A}} X \otimes (M \otimes A) \xrightarrow{\mathrm{id}_X \otimes \rho_M} X \otimes M.$$

Similarly, ${}_A\mathcal{C}$ is a right $\mathcal{C}$-module category.

Proposition 2.15. [30, Remark 3.1] [8, Proposition 7.8.30] *We have that $\mathcal{C}_A$ is an indecomposable (resp., semisimple) $\mathcal{C}$-module category if A is an indecomposable (resp., separable) algebra in $\mathcal{C}$.* $\square$

We have an equivalent definition of a Frobenius algebra in a (rigid) monoidal category, which uses the terminology above.

Remark 2.16. Let $\mathcal{C}$ be a rigid monoidal category. Then ${}^*A$ is a left A-module via

$$\lambda_{{}^*A} := (\mathrm{id}_{{}^*A} \otimes \mathsf{ev}'_A)\alpha_{{}^*A,A,{}^*A}(\mathrm{id}_{{}^*A} \otimes m_A \otimes \mathrm{id}_{{}^*A})(\alpha_{{}^*A,A,A} \otimes \mathrm{id}_{{}^*A})(\mathsf{coev}'_A \otimes \mathrm{id}_{A{}^*A})(l_A^{-1} \otimes \mathrm{id}_{A{}^*A}).$$

Now by [17], a Frobenius algebra in $\mathcal{C}$ can be equivalently defined as an algebra A in $\mathcal{C}$ so that (A, λ_A) is isomorphic to $({}^*A, \lambda_{{}^*A})$ as left A-modules.

Next, we turn our attention to Morita equivalence of algebras in monoidal categories.

Definition 2.17. We say that two algebras A and B in $\mathcal{C}$ are *Morita equivalent (in $\mathcal{C}$)* if $\mathcal{C}_A \sim \mathcal{C}_B$ as (left) $\mathcal{C}$-module categories.

Several algebraic properties are preserved under Morita equivalence, such as indecomposability and separability. We will discuss a characterization of Morita equivalence in terms of bimodules in Section 2.5.

2.4. **Categories of bimodules over algebras.** Fix a monoidal category $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$. We recall here preliminary notions on bimodules over algebras in $\mathcal{C}$. For general information, see [25, Section 3.3] and [8, Section 7.8].

Definition 2.18 (${}_A\mathcal{C}_A$)**.** Take $A := (A, m_A, u_A) \in \mathsf{Alg}(\mathcal{C})$. An *$A$-bimodule in $\mathcal{C}$* is a triple (M, λ_M, ρ_M), where $M \in \mathcal{C}$, and $\lambda_M : A \otimes M \to M$ and $\rho_M : M \otimes A \to M$ are morphisms in $\mathcal{C}$, so that $(M, \lambda_M) \in {}_A\mathcal{C}$ and $(M, \rho_M) \in \mathcal{C}_A$ with

$$\lambda_M(\mathrm{id}_A \otimes \rho_M)\alpha_{A,M,A} \;=\; \rho_M(\lambda_M \otimes \mathrm{id}_A).$$

A *morphism* of A-bimodules in $\mathcal{C}$ is a morphism $f : M \to N$ in $\mathcal{C}$ that is simultaneously a morphism in both ${}_A\mathcal{C}$ and $\mathcal{C}_A$. Bimodules over A in $\mathcal{C}$ and their morphisms form a category, which we denote by ${}_A\mathcal{C}_A$.

Definition 2.19 ($\otimes_A$, $\pi_{M,N}, \pi^A_{M,N}$)**.** Take A-bimodules M and N in $\mathcal{C}$. The *tensor product* of M and N over A is the object of ${}_A\mathcal{C}_A$ given by

$$M \otimes_A N := \mathrm{coker}\big(\rho_M \otimes \mathrm{id}_N - (\mathrm{id}_M \otimes \lambda_N)\alpha_{M,A,N}\big).$$

Let $\pi_{M,N} := \pi^A_{M,N} : M \otimes N \to M \otimes_A N$ denote the canonical projection, a morphism in $\mathcal{C}$. Moreover, $M \otimes_A N$ is an A-bimodule via morphisms:

$$\lambda_{M\otimes_A N} : A \otimes (M \otimes_A N) \to M \otimes_A N \quad \text{and} \quad \rho_{M\otimes_A N} : (M \otimes_A N) \otimes A \to M \otimes_A N$$

so that

$$\lambda_{M\otimes_A N}(\mathrm{id}_A \otimes \pi_{M,N}) = \pi_{M,N}(\lambda_M \otimes \mathrm{id}_N)\alpha^{-1}_{A,M,N},$$
$$\pi_{M,N}(\mathrm{id}_M \otimes \rho_N)\alpha_{M,N,A} = \rho_{M\otimes_A N}(\pi_{M,N} \otimes \mathrm{id}_A).$$

Proposition 2.20 ((${}_A\mathcal{C}_A$, $\otimes_A$, A, $\alpha^A_{*,*,*}$, l^A_*, r^A_*))**.** [25, Section 3.3.2] *The category ${}_A\mathcal{C}_A$ has the structure of a monoidal category with*

- *tensor product $\otimes_A$,*
- *unit object A, and*
- *associativity constraint $\alpha^A_{X,X',X''} : (X \otimes_A X') \otimes_A X'' \xrightarrow{\sim} X \otimes_A (X' \otimes_A X'')$ for $X, X', X'' \in \mathcal{C}$, so that*

$$\alpha^A_{X,X',X''}\, \pi_{X\otimes_A X',X''}\, (\pi_{X,X'} \otimes \mathrm{id}_{X''}) = \pi_{X,X'\otimes_A X''}\, (\mathrm{id}_X \otimes \pi_{X',X''})\, \alpha_{X,X',X''},$$

- *unit constraints $l^A_X : A \otimes_A X \xrightarrow{\sim} X$ and $r^A_X : X \otimes_A A \xrightarrow{\sim} X$ so that*

$$l^A_X\, \pi_{A,X} = \lambda_X \qquad \textit{and} \qquad r^A_X\, \pi_{X,A} = \rho_X. \qquad \square$$

Moreover, for maps $f : X \to W$ and $g : Y \to Z$ in ${}_A\mathcal{C}_A$, we get that

$$(f \otimes_A g)\, \pi_{X,Y} \;=\; \pi_{W,Z}\, (f \otimes g) \tag{2.21}$$

as maps in $\mathcal{C}$.

On the other hand, [35, Section 5] and [12] discuss conditions on A to yield that the category of bimodules ${}_A\mathcal{C}_A$ above is rigid. We ask, in general:

Question 2.22. Given a (rigid) monoidal category $\mathcal{C}$, what precise conditions on $A \in \mathsf{Alg}(\mathcal{C})$ need to be imposed to get that the category of bimodules ${}_A\mathcal{C}_A$ is rigid?

2.5. **On Morita equivalence of algebras.** We provide here characterizations for the Morita equivalence of algebras in monoidal categories [Definition 2.17], and provide other preliminary results that we will need later in Section 4. First, consider the following notation.

Definition 2.23 ($\overline{\alpha}_{*,*,*}$)**.** Let $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ be a monoidal category, and take two algebras A and B in $\mathcal{C}$. Let $X, Z \in {}_A\mathcal{C}_B$ and $Y \in {}_B\mathcal{C}_A$. Take

$$\overline{\alpha}_{X,Y,Z} : (X \otimes_B Y) \otimes_A Z \to X \otimes_B (Y \otimes_A Z)$$

to be the morphism in $\mathcal{C}$ defined by the commutative diagram:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\pi^B_{X,Y} \otimes \mathrm{id}_Z"] \arrow[d, "\alpha"'] & (X \otimes_B Y) \otimes Z \arrow[r, "\pi^A_{XY,Z}"] & (X \otimes_B Y) \otimes_A Z \arrow[d, dashed, "\overline{\alpha}"] \\
X \otimes (Y \otimes Z) \arrow[r, "\pi^B_{X,YZ}"] & X \otimes_B (Y \otimes Z) \arrow[r, "\mathrm{id}_X \otimes_B \pi^A_{Y,Z}"] & X \otimes_B (Y \otimes_A Z).
\end{tikzcd}$$

The same notation will apply in the case when the roles A and B are reversed.

Lemma 2.24. [8, Exercise 7.8.28] *The morphism $\overline{\alpha}$ exists, and is an isomorphism in $\mathcal{C}$.* □

Proposition 2.25. *Let $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ be a monoidal category, and take two algebras A and B in $\mathcal{C}$. Then the following statements hold.*

(a) *A and B are Morita equivalent if and only if there exist bimodules $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_A$ so that $P \otimes_B Q \cong A$ in ${}_A\mathcal{C}_A$ and $Q \otimes_A P \cong B$ in ${}_B\mathcal{C}_B$.*

(b) *If there exist bimodules $P \in {}_A\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_A$ along with epimorphisms*

$$\tau : P \otimes_B Q \twoheadrightarrow A \text{ in } {}_A\mathcal{C}_A \qquad \text{and} \qquad \mu : Q \otimes_A P \twoheadrightarrow B \text{ in } {}_B\mathcal{C}_B$$

so that the diagrams $(*)$ and $(**)$ below commute in $\mathcal{C}$, then the equivalent conditions of part (a) hold.

$$\begin{tikzcd}
(P \otimes_B Q) \otimes_A P \arrow[rr, "\overline{\alpha}"] \arrow[d, two heads, "\tau \otimes_A \mathrm{id}_P"] & & P \otimes_B (Q \otimes_A P) \arrow[d, two heads, "\mathrm{id}_P \otimes_B \mu"'] \\
A \otimes_A P \arrow[dr, "l^A_P"'] & (*) & P \otimes_B B \arrow[dl, "r^B_P"] \\
& P &
\end{tikzcd}
\qquad
\begin{tikzcd}
(Q \otimes_A P) \otimes_B Q \arrow[rr, "\overline{\alpha}"] \arrow[d, two heads, "\mu \otimes_B \mathrm{id}_Q"] & & Q \otimes_A (P \otimes_B Q) \arrow[d, two heads, "\mathrm{id}_Q \otimes_A \tau"'] \\
B \otimes_B Q \arrow[dr, "l^B_Q"'] & (**) & Q \otimes_A A \arrow[dl, "r^A_Q"] \\
& Q &
\end{tikzcd}$$

Proof. (a) This is well-known; see, e.g., [30, Remark 3.2] and [14].

(b) Since $\mathcal{C}$ is assumed to be abelian, the category ${}_A\mathcal{C}_A$ is also abelian (see, e.g., [8, Exercise 7.8.7]). So it suffices to show τ and μ are monomorphisms in ${}_A\mathcal{C}_A$ as epic monomorphisms are isomorphisms in abelian categories. We prove the statement for τ; the proof for μ will follow similarly.

Take morphisms $g_1, g_2 : W \to P \otimes_B Q$ in ${}_A\mathcal{C}_A$ so that $\tau g_1 = \tau g_2$ as morphisms $W \to A$ in ${}_A\mathcal{C}_A$. Consider the following commutative diagram in $\mathcal{C}$, where we suppress the $\otimes_*$ symbol in morphisms. We also invoke Lemma 2.24 in all of the diagrams below for the existence of the morphism $\overline{\alpha}$.

$$\begin{tikzcd}
(W \otimes_A P) \otimes_B Q \arrow[rrr, "\overline{\alpha}"] \arrow[ddd, "g_i \text{ id id}"'] & & & W \otimes_A (P \otimes_B Q) \arrow[r, "\text{id}\tau"] \arrow[d, "g_i \text{ id id}"'] & W \otimes_A A \arrow[ddd, "g_i \text{ id}"] \\
& & & (P \otimes_B Q) \otimes_A (P \otimes_B Q) \arrow[d, "\overline{\alpha}"] \arrow[ddr, "\text{id id } \tau"] & \\
& (P \otimes_B (Q \otimes_A P)) \otimes_B Q \arrow[dr, "\overline{\alpha}"] \arrow[dd, "\text{id } \mu \text{ id}"] & & P \otimes_B (Q \otimes_A (P \otimes_B Q)) \arrow[dl, "\text{id } \overline{\alpha}^{-1}"'] & \\
((P \otimes_B Q) \otimes_A P) \otimes_B Q \arrow[ur, "\overline{\alpha} \text{ id}"] \arrow[dd, "\tau \text{ id id}"'] & & P \otimes_B ((Q \otimes_A P) \otimes_B Q) \arrow[dd, "\text{id } \mu \text{ id}"'] \arrow[dr, "\text{id } \overline{\alpha}"] & & (P \otimes_B Q) \otimes_A A \arrow[dd, "\overline{\alpha}"] \\
& (*) \; (P \otimes_B B) \otimes_B Q \arrow[dr, "\overline{\alpha}"] \arrow[ddr, "r_P^B \text{ id}"'] & & P \otimes_B (Q \otimes_A (P \otimes_B Q)) \arrow[dr, "\text{id id } \tau"] & \\
(A \otimes_A P) \otimes_B Q \arrow[dr, "l_P^A \text{ id}"'] & & P \otimes_B (B \otimes_B Q) \arrow[d, "\text{id } l_Q^B"] & (**) & P \otimes_B (Q \otimes_A A) \arrow[dll, "\text{id } r_Q^A"] \\
& & P \otimes_B Q & &
\end{tikzcd}$$

Now with the diagram commuting, the assumption $\tau g_1 = \tau g_2$ implies that

$$\begin{aligned}
&(\mathrm{id}_P \otimes_B r_Q^A)\, \overline{\alpha}_{P,Q,A}\, (g_1 \otimes_A \mathrm{id}_A)(\mathrm{id}_W \otimes_A \tau)\, \overline{\alpha}_{W,P,Q} \\
&= (\mathrm{id}_P \otimes_B r_Q^A)\, \overline{\alpha}_{P,Q,A}\, (g_2 \otimes_A \mathrm{id}_A)(\mathrm{id}_W \otimes_A \tau)\, \overline{\alpha}_{W,P,Q}.
\end{aligned}$$

Since $\overline{\alpha}$ is an epimorphism by Lemma 2.24, and $(\mathrm{id}_W \otimes_A \tau)$ is also an epimorphism by assumption, we get that

$$(\mathrm{id}_P \otimes_B r_Q^A)\, \overline{\alpha}_{P,Q,A}\, (g_1 \otimes_A \mathrm{id}_A) = (\mathrm{id}_P \otimes_B r_Q^A)\, \overline{\alpha}_{P,Q,A}\, (g_2 \otimes_A \mathrm{id}_A).$$

By [8, Exercise 7.8.22] we have that r_Q^A is an isomorphism in ${}_B\mathcal{C}$, so $\mathrm{id}_P \otimes_B r_Q^A$ is an isomorphism in $\mathcal{C}$ as well. Therefore,

$$\overline{\alpha}_{P,Q,A}\, (g_1 \otimes_A \mathrm{id}_A) = \overline{\alpha}_{P,Q,A}\, (g_2 \otimes_A \mathrm{id}_A).$$

Finally, by Lemma 2.24, $\overline{\alpha}$ is an isomorphism. Thus, $(g_1 \otimes_A \mathrm{id}_A) = (g_2 \otimes_A \mathrm{id}_A)$, and $g_1 = g_2$, as desired. □

Part (a) is a generalization of a classical ring theory result, which is presented, e.g., in [5, Theorem 4.4.5]. The proof of (b) is a generalization of [5, Lemma 4.5.2]. Moreover, the result below generalizes the classic result that a $\Bbbk$-algebra R is Morita equivalent to a matrix algebra $\mathrm{Mat}_n(R)$ over $\Bbbk$. (Indeed, the classical result is recovered from the following result by letting $\mathcal{C}$ be the monoidal category of finite-dimensional $\Bbbk$-vector spaces with $S = R$ and $V = \Bbbk^{\oplus n}$.)

Proposition 2.26. *Let $\mathcal{C}$ be a monoidal category, and take an algebra S in $\mathcal{C}$ and an object V in $\mathcal{C}$ so that it has a right dual in $\mathcal{C}$. Then,*

(a) $({}^*V \otimes S) \otimes V \in \mathsf{Alg}(\mathcal{C})$ *with*

$$m_{({}^*V \otimes S) \otimes V} = (\mathrm{id}_{{}^*V} \otimes m_S \otimes \mathrm{id}_V)\alpha_2(r_{{}^*V \otimes S} \otimes \mathrm{id}_{SV})(\mathrm{id}_{{}^*VS} \otimes \mathsf{ev}'_V \otimes \mathrm{id}_{SV})\alpha_1, \ \textit{for}$$

$$\begin{aligned}
\alpha_1 &= \alpha^{-1}_{{}^*VS,V{}^*V,SV}(\mathrm{id}_{{}^*VS} \otimes \alpha^{-1}_{V,{}^*V,SV})(\mathrm{id}_{{}^*VSV} \otimes \alpha_{{}^*V,S,V})\alpha_{{}^*VS,V,{}^*VSV}, \\
\alpha_2 &= (\alpha_{{}^*V,S,S} \otimes \mathrm{id}_V)\alpha^{-1}_{{}^*VS,S,V}, \ \textit{and}
\end{aligned}$$

$$u_{({}^*V \otimes S) \otimes V} = (\mathrm{id}_{{}^*V} \otimes u_S \otimes \mathrm{id}_V)(r^{-1}_{{}^*V} \otimes \mathrm{id}_V)\mathsf{coev}'_V;$$

(b) *S and $({}^*V \otimes S) \otimes V$ are Morita equivalent as algebras in $\mathcal{C}$.*

Proof. (a) The algebra axioms hold in the strict case by the following commutative diagrams. We leave the non-strict case to the reader.

$$\begin{tikzcd}
{}^*VSV{}^*VSV{}^*VSV \arrow[r, "\mathrm{id}_{..}\,\mathsf{ev}'_V\,\mathrm{id}_{.....}"] \arrow[d, "\mathrm{id}_{.....}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"] \arrow[rr, bend left, "m\,\mathrm{id}_{...}"] \arrow[dd, bend right=60, "\mathrm{id}_{...}\,m"'] & {}^*VSSV{}^*VSV \arrow[r, "\mathrm{id}\,m_S\,\mathrm{id}_{....}"] \arrow[d, "\mathrm{id}_{...}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"'] & {}^*VSV{}^*VSV \arrow[d, "\mathrm{id}_{..}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"'] \arrow[dd, bend left=60, "m"] \\
{}^*VSV{}^*VSSV \arrow[r, "\mathrm{id}_{..}\,\mathsf{ev}'_V\,\mathrm{id}_{...}"] \arrow[d, "\mathrm{id}_{....}\,m_S\,\mathrm{id}"] & {}^*VSSSV \arrow[r, "\mathrm{id}\,m_S\,\mathrm{id}_{..}"] \arrow[d, "\mathrm{id}_{..}\,m_S\,\mathrm{id}"'] & {}^*VSSV \arrow[d, "\mathrm{id}\,m_S\,\mathrm{id}"'] \\
{}^*VSV{}^*VSV \arrow[r, "\mathrm{id}_{..}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"] \arrow[rr, bend right, "m"'] & {}^*VSSV \arrow[r, "\mathrm{id}\,m_S\,\mathrm{id}"] & {}^*VSV
\end{tikzcd}$$

In particular, the bottom right square commutes due to the associativity of m_S.

$$\begin{tikzcd}
& & & {}^*VSV{}^*VSV \arrow[d, "\mathrm{id}_{..}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"'] \arrow[dd, bend left=60, "m"] \\
& {}^*VV{}^*VSV \arrow[r, "\mathrm{id}\,\mathsf{ev}'_V\,\mathrm{id}_{..}"] \arrow[urr, bend left, "\mathrm{id}\,u_S\,\mathrm{id}_{....}"] & {}^*VSV \arrow[r, "\mathrm{id}\,u_S\,\mathrm{id}_{..}"] \arrow[dr, equal] & {}^*VSSV \arrow[d, "\mathrm{id}\,m_S\,\mathrm{id}"'] \\
{}^*VSV \arrow[ur, "\mathsf{coev}'_V\,\mathrm{id}_{...}"'] \arrow[urr, equal] \arrow[uurrr, bend left, "u\,\mathrm{id}_{...}"] & & & {}^*VSV
\end{tikzcd}$$

Here, the bottom triangles commute due to the rigidity and unit axioms for S. The other unit axiom for $({}^*V \otimes S) \otimes V$ holds likewise.

(b) Let T denote the algebra $({}^*V \otimes S) \otimes V$ in part (a). Let $P := {}^*V \otimes S$ and $Q := S \otimes V$. It follows from the associativity of m_S, and naturality of $\alpha_{*,*,*}$ and r_*, that the morphisms

$$\begin{aligned}
\lambda_P^T &= (\mathrm{id}_{^*V} \otimes m_S)(\mathrm{id}_{^*VS} \otimes r_S \otimes \mathrm{id}_S)(\mathrm{id}_{^*VS} \otimes \mathsf{ev}'_V \otimes \mathrm{id}_S)\alpha_3 \\
\rho_P^S &= (\mathrm{id}_{^*V} \otimes m_S)\alpha_{^*V,S,S} \\
\lambda_Q^S &= (m_S \otimes \mathrm{id}_V)\alpha^{-1}_{S,S,V} \\
\rho_Q^T &= (m_S \otimes \mathrm{id}_V)(r_S \otimes \mathrm{id}_{SV})(\mathrm{id}_S \otimes \mathsf{ev}'_V \otimes \mathrm{id}_{SV})\alpha_4,
\end{aligned}$$

for

$$\begin{aligned}
\alpha_3 &= (\mathrm{id}_{^*V} \otimes \alpha_{S,V,^*V} \otimes \mathrm{id}_S)(\mathrm{id}_{^*V} \otimes \alpha^{-1}_{SV,^*V,S})\alpha_{^*V,SV,^*VS}(\alpha_{^*V,S,V} \otimes \mathrm{id}_{^*VS}) \\
\alpha_4 &= (\alpha_{S,V,^*V} \otimes \mathrm{id}_{SV})(\alpha^{-1}_{SV,^*V,S} \otimes \mathrm{id}_V)\alpha^{-1}_{SV,^*VS,V},
\end{aligned}$$

imply that $(P, \lambda_P^T, \rho_P^S) \in {}_T\mathcal{C}_S$ and $(Q, \lambda_Q^S, \rho_Q^T) \in {}_S\mathcal{C}_T$. Moreover, consider the morphisms

$$\begin{aligned}
\widehat{\tau} &= (\mathrm{id}_{^*V} \otimes m_S \otimes \mathrm{id}_V)(\alpha_{^*V,S,S} \otimes \mathrm{id}_V)\alpha^{-1}_{^*VS,S,V} &&: P \otimes Q \to T, \\
\widehat{\mu} &= m_S(r_S \otimes \mathrm{id}_S)(\mathrm{id}_S \otimes \mathsf{ev}'_V \otimes \mathrm{id}_S)(\alpha_{S,V,^*V} \otimes \mathrm{id}_S)\alpha^{-1}_{SV,^*V,S} &&: Q \otimes P \to S.
\end{aligned}$$

It follows from the associativity of m_S, and naturality of $\alpha_{*,*,*}$ and r_*, that $\widehat{\tau} \in {}_T\mathcal{C}_T$ and $\widehat{\mu} \in {}_S\mathcal{C}_S$. It is also clear that $\widehat{\tau}$ and $\widehat{\mu}$ are epimorphisms in $\mathcal{C}$. Moreover, the morphisms factor through epimorphisms $\tau : P \otimes_S Q \to T$ and $\mu : Q \otimes_T P \to S$, respectively, so that $\widehat{\tau} = \tau\, \pi^S_{P,Q}$ and $\widehat{\mu} = \mu\, \pi^T_{Q,P}$. Indeed, by the naturality of $\alpha_{*,*,*}$

and the associativity of m_S, we get that $\widehat{\tau}(\rho^S_P \otimes \mathrm{id}_Q) = \widehat{\tau}(\mathrm{id}_P \otimes \lambda^S_Q)\alpha_{P,S,Q}$. So the claim for τ follows from the definition of $P \otimes_S Q$. Likewise, the claim for μ holds.

Finally, by Proposition 2.25(b), it suffices to show that τ and μ satisfy the diagrams $(*)$ and $(**)$ there. We will do so for $(*)$ in the strict case, and the general case, along with $(**)$ will hold in a similar manner. The unadorned $\otimes$ symbol in morphisms are suppressed below.

$$\begin{tikzcd}
(P \otimes_S Q) \otimes_T P \arrow[rrrr, "\overline{\alpha}"] \arrow[ddddd, "\tau \otimes_T \mathrm{id}_P"] & & & & P \otimes_S (Q \otimes_T P) \arrow[ddddd, "\mathrm{id}_P \otimes_S \mu"] \\
& (P \otimes_S Q) \otimes P \arrow[ul, "\pi^T_{P \otimes_S Q, P}"'] \arrow[ddd, "\tau\,\mathrm{id}_P"'] & P \otimes_S (Q \otimes P) \arrow[urr, "\mathrm{id}_P \otimes_S \pi^T_{Q,P}"] & P \otimes (Q \otimes_T P) \arrow[ur, "\pi^S_{P, Q \otimes_T P}"'] \arrow[ddd, "\mathrm{id}_P \mu"] & \\
& & P \otimes Q \otimes P \arrow[ul, bend right, "\pi^S_{P,Q}\mathrm{id}_P"] \arrow[u, "\pi^S_{P, Q \otimes P}"'] \arrow[ur, bend left, "\mathrm{id}_P \pi^T_{Q,P}"'] \arrow[ddl, bend right, "\widehat{\tau}\mathrm{id}_P"'] \arrow[d, bend left, "\mathrm{id}_{{}^*VSS}\mathsf{ev}'_V\mathrm{id}_S"'] \arrow[ddr, bend left, "\mathrm{id}_P \widehat{\mu}"] & & \\
& & {}^*VSSS \arrow[d, "\mathrm{id}_{{}^*V} m_S \mathrm{id}_S"'] \arrow[dr, "\mathrm{id}_{{}^*VS} m_S"] & & \\
& T \otimes P \arrow[r, "\mathrm{id}_{{}^*VS}\mathsf{ev}'_V\mathrm{id}_S"] \arrow[dl, "\pi^T_{T,P}"'] \arrow[ddr, "\lambda^T_P"] & {}^*VSS \arrow[d, "\mathrm{id}_{{}^*VS} m_S"] \arrow[dd, bend right, "\mathrm{id}_{{}^*V} m_S"'] & P \otimes S \arrow[dl, "\mathrm{id}_{{}^*V} m_S"'] \arrow[ddl, "\rho^S_P"'] \arrow[dr, "\pi^S_{P,S}"] & \\
T \otimes_T P \arrow[drr, "l^T_P"'] & & {}^*VS & & P \otimes_S S \arrow[dll, "r^S_P"] \\
& & P & &
\end{tikzcd}$$

All regions commute either by the definitions of the maps involved, by (2.21), or by the associativity of m_S. $\square$

We will use the techniques in this proposition in the proof of Theorem 4.9 below, and the algebra in part (a) above is mentioned in Remark 7.5 below.

3. A Frobenius monoidal functor Φ to a category of bimodules

In this section, take $\mathcal{C}$ to be a monoidal category. Our main result in this section is that, when A is a special Frobenius algebra in $\mathcal{C}$, we establish on the 'free' functor from $\mathcal{C}$ to the category of A-bimodules in $\mathcal{C}$ a Frobenius monoidal structure. To begin, consider the notation below.

Notation 3.1 ($\tilde{*}$)**.** Take $A \in \mathsf{Alg}(\mathcal{C})$, and take objects $X, X', W \in {}_A\mathcal{C}_A$. For a map $f : X \otimes_A X' \to W$ in ${}_A\mathcal{C}_A$, let $\tilde{f} : X \otimes X' \to W$ denote its lift in $\mathcal{C}$ in the sense that $\tilde{f} = f\ \pi_{X,X'}$.

Now, we have the following result.

Theorem 3.2 (Φ)**.** *Take $\mathcal{C} := (\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ to be a monoidal category and let $A = (A, m, u, \Delta, \varepsilon)$ be a special Frobenius algebra in $\mathcal{C}$. Then the following functor is Frobenius monoidal:*

$$\Phi : \mathcal{C} \to {}_A\mathcal{C}_A$$

$$\begin{aligned} X &\mapsto (A \otimes X) \otimes A && \textit{(as objects)} \\ \varphi &\mapsto (\mathrm{id}_A \otimes \varphi) \otimes \mathrm{id}_A && \textit{(as morphisms).} \end{aligned}$$

Here, the monoidal structure $\Phi_{X,X'}$ is defined by the lift of $\widetilde{\Phi}_{X,X'}$, that is,

$$\widetilde{\Phi}_{X,X'} = \Phi_{X,X'}\ \pi_{\Phi(X),\Phi(X')},$$

with:

$$\begin{aligned} \widetilde{\Phi}_{X,X'} = {} & (\alpha_{A,X,X'} \otimes \mathrm{id}_A)(\mathrm{id}_{AX} \otimes\ l_{X'} \otimes \mathrm{id}_A)(\mathrm{id}_{AX} \otimes \varepsilon_A m_A \otimes \mathrm{id}_{X',A})\underline{\alpha}, \quad \textit{for} \\ \underline{\alpha} := {} & (\mathrm{id}_{AX} \otimes \alpha^{-1}_{A,A,X'} \otimes \mathrm{id}_A)(\alpha_{AX,A,AX'} \otimes \mathrm{id}_A)\alpha^{-1}_{AXA,AX',A}, \end{aligned}$$

and by

$$\Phi_0 = (r_A^{-1} \otimes \mathrm{id}_A)\Delta_A.$$

Moreover, the comonoidal structure $\Phi^{X,X'} = \pi_{\Phi(X),\Phi(X')}\ \widetilde{\Phi}^{X,X'}$ is given by

$$\begin{aligned} \widetilde{\Phi}^{X,X'} = {} & \underline{\alpha'}(\mathrm{id}_{AX} \otimes \Delta_A u_A \otimes \mathrm{id}_{X',A})(\mathrm{id}_{A,X} \otimes l^{-1}_{X'} \otimes \mathrm{id}_A)(\alpha^{-1}_{A,X,X'} \otimes \mathrm{id}_A), \quad \textit{for} \\ \underline{\alpha'} := {} & \alpha_{AXA,AX',A}(\alpha^{-1}_{AX,A,AX'} \otimes \mathrm{id}_A)(\mathrm{id}_{AX} \otimes \alpha_{A,A,X'} \otimes \mathrm{id}_A), \end{aligned}$$

and by

$$\Phi^0 = m_A(r_A \otimes \mathrm{id}_A).$$

Proof. We need to verify the following conditions:

(a) $\Phi(X)$ is an A-bimodule in $\mathcal{C}$;
(b) $\Phi_{X,X'}$ is well defined via $\widetilde{\Phi}_{X,X'}$, that is,
  (b.1) $\widetilde{\Phi}_{X,X'}\ (\rho^A_{\Phi(X)} \otimes \mathrm{id}_{\Phi(X')}) = \widetilde{\Phi}_{X,X'}\ (\mathrm{id}_{\Phi(X)} \otimes \lambda^A_{\Phi(X')})\ \alpha_{\Phi(X),A,\Phi(X')}$, and
  (b.2) $\Phi_{X,X'}$ is an A-bimodule map;
(c) Φ_0 is an A-bimodule map;
(d) $\Phi^{X,X'}$ is an A-bimodule map;
(e) Φ^0 is an A-bimodule map;
(f) the associativity condition:

$$\begin{aligned} &\Phi_{XX',X''}\ (\Phi_{X,X'} \otimes_A \mathrm{id}_{\Phi(X'')}) \\ &= \Phi(\alpha^{-1}_{X,X',X''})\ \Phi_{X,X'X''}\ (\mathrm{id}_{\Phi(X)} \otimes_A \Phi_{X',X''})\ \alpha^A_{\Phi(X),\Phi(X'),\Phi(X'')}; \end{aligned}$$

(g) the unitality conditions:

$$\begin{aligned} l^A_{\Phi(X)} &= \Phi(l_X)\ \Phi_{\mathbb{1},X}\ (\Phi_0 \otimes_A \mathrm{id}_{\Phi(X)}), \\ r^A_{\Phi(X)} &= \Phi(r_X)\ \Phi_{X,\mathbb{1}}\ (\mathrm{id}_{\Phi(X)} \otimes_A \Phi_0); \end{aligned}$$

(h) the coassociativity condition:

$$\begin{aligned} &\alpha^A_{\Phi(X),\Phi(X'),\Phi(X'')}\ (\Phi^{X,X'} \otimes_A \mathrm{id}_{\Phi(X'')})\ \Phi^{XX',X''} \\ &= (\mathrm{id}_{\Phi(X)} \otimes_A \Phi^{X',X''})\ \Phi^{X,X'X''}\ \Phi(\alpha_{X,X',X''}); \end{aligned}$$

(i) the counitality conditions:

$$\begin{aligned} l^A_{\Phi(X)}\ (\Phi^0 \otimes_A \mathrm{id}_{\Phi(X)})\ \Phi^{\mathbb{1},X} &= \Phi(l_X), \\ r^A_{\Phi(X)}\ (\mathrm{id}_{\Phi(X)} \otimes_A \Phi^0)\ \Phi^{X,\mathbb{1}} &= \Phi(r_X); \quad \textit{and} \end{aligned}$$

(j) the Frobenius conditions:

$$
\begin{aligned}
&(\Phi_{X,X'} \otimes_A \mathrm{id}_{\Phi(X'')})\, (\alpha^A_{\Phi(X),\Phi(X'),\Phi(X'')})^{-1}\, (\mathrm{id}_{\Phi(X)} \otimes_A \Phi^{X',X''}) \\
&= \Phi^{XX',X''}\, \Phi(\alpha^{-1}_{X,X\prime,X''})\, \Phi_{X,X'X''}, \\
&(\mathrm{id}_{\Phi(X)} \otimes_A \Phi_{X',X''})\, \alpha^A_{\Phi(X),\Phi(X'),\Phi(X'')}\, (\Phi^{X,X'} \otimes_A \mathrm{id}_{\Phi(X'')}) \\
&= \Phi^{X,X'X''}\, \Phi(\alpha_{X,X',X''})\, \Phi_{XX',X''}.
\end{aligned}
$$

We provide some details here, but most of the details will be left to the reader. Note that in the diagrams below, we will omit the $\otimes$ symbol in the nodes and arrows, and also omit parentheses in the arrows, to make them more compact.

(a) The right and left A-module structure of $\Phi(X) = (A \otimes X) \otimes A$ are given by

$$
\begin{aligned}
\rho^A_{\Phi(X)} &:= (\mathrm{id}_{AX} \otimes m_A)\, \alpha_{AX,A,A}, \\
\lambda^A_{\Phi(X)} &:= ((m_A \otimes \mathrm{id}_X)\, \alpha^{-1}_{A,A,X} \otimes \mathrm{id}_A)\, \alpha^{-1}_{A,AX,A}
\end{aligned}
$$

respectively. We leave the details for the verification of the left A-module condition, right A-module structure and the A-bimodule compatibility to the reader.

(b.1) Let us see that

$$\widetilde{\Phi}_{X,X'}(\rho_{\Phi(X)} \otimes \mathrm{id}_{\Phi(X')}) = \widetilde{\Phi}_{X,X'}(\mathrm{id}_{\Phi(X)} \otimes \lambda_{\Phi(X')})\alpha_{\Phi(X),A,\Phi(X')}$$

in the strict case. Namely, the following diagram commutes.

$$
\begin{tikzcd}
AXAAAX'A \arrow[rr, "\mathrm{id}_{AX} m_A \mathrm{id}_{AX'A}"] \arrow[dd, "\mathrm{id}_{AXA} m_A \mathrm{id}_{X'A}"'] & & AXAAX'A \arrow[d, "\mathrm{id}_{AX} m_A \mathrm{id}_{X'A}"] \\
 & (1) & AXAX'A \arrow[d, "\mathrm{id}_{AX} \varepsilon_A \mathrm{id}_{X'A}"] \\
AXAAX'A \arrow[r, "\mathrm{id}_{AX} m_A \mathrm{id}_{X'A}"] & AXAX'A \arrow[r, "\mathrm{id}_{AX} \varepsilon_A \mathrm{id}_{X'A}"] \arrow[ur, equal] & AXX'A
\end{tikzcd}
$$

Here, (1) commutes because m_A is associative. Therefore, there is a unique map $\Phi_{X,X'} : \Phi(X) \otimes_A \Phi(X') \to \Phi(X \otimes X')$ such that $\widetilde{\Phi}_{X,X'} = \Phi_{X,X'}\ \pi_{\Phi(X),\Phi(X')}$.

(b.2) Let us prove that $\Phi_{X,X'}$ is a right A-module map when $\mathcal{C}$ is strict. The proof for the rest of the part, including the non-strict case, is left to the reader. Consider the following diagram.

$$
\begin{tikzcd}
(\Phi(X) \otimes_A \Phi(X'))A \arrow[rrrr, "\Phi_{X,X'} \mathrm{id}_A"] \arrow[ddd, "\rho_{\Phi(X) \otimes_A \Phi(X')}"'] & & (1) & & \Phi(XX')A \arrow[ddd, "\rho_{\Phi(XX')}"] \\
 & (2) & \Phi(X)\Phi(X')A \arrow[ull, "\pi_{\Phi(X),\Phi(X')} \mathrm{id}_A"'] \arrow[urr, "\widetilde{\Phi}_{X,X'} \mathrm{id}_A"] \arrow[d, "\mathrm{id}_{\Phi(X)} \rho_{\Phi(X')}"'] & (3) & \\
 & & \Phi(X)\Phi(X') \arrow[dll, "\pi_{\Phi(X),\Phi(X')}"'] \arrow[drr, "\widetilde{\Phi}_{X,X'}"] & & \\
\Phi(X) \otimes_A \Phi(X') \arrow[rrrr, "\Phi_{X,X'}"'] & & (4) & & \Phi(XX')
\end{tikzcd}
$$

We have that (1) and (4) commute by the definition of $\widetilde{\Phi}_{X,X'}$, and (2) commutes by the definition of $\rho_{\Phi(X)\otimes_A\Phi(X')}$. Moreover, (3) commutes via the following diagram:

$$\begin{tikzcd}
AXAAX'AA \arrow[r, "\mathrm{id}_{AX}m_A\mathrm{id}_{X'AA}"] \arrow[d, "\mathrm{id}_{AXAAX'}m_A"] & AXAX'AA \arrow[r, "\mathrm{id}_{AX}\varepsilon_A\mathrm{id}_{X'AA}"] \arrow[d, "\mathrm{id}_{AXAX'}m_A"] & AXX'AA \arrow[d, "\mathrm{id}_{AXX'}m_A"] \\
AXAAX'A \arrow[r, "\mathrm{id}_{AX}m_A\mathrm{id}_{X'A}"] & AXAX'A \arrow[r, "\mathrm{id}_{AX}\varepsilon_A\mathrm{id}_{X'A}"] & AXX'A;
\end{tikzcd}$$

each square commutes as a result of the maps being applied in different slots.

(c) We get that Φ_0 is a right A-module map when $\mathcal{C}$ is strict because A is Frobenius, and we leave the rest to the reader.

(d) – (h) We leave these details to the reader.

(i) We have that Φ^0 satisfies the counitality condition when A is special as follows. We check the left counitality constraint for $\mathcal{C}$ strict, and leave the rest to the reader. In the following diagram

$$\begin{tikzcd}
\Phi(\mathbb{1})\otimes_A\Phi(X) \arrow[rrr, "\Phi^0\otimes_A\mathrm{id}_{\Phi(X)}"] & & & A\otimes_A\Phi(X) \arrow[d, "l^A_{\Phi(X)}"] \\
(1) & \Phi(\mathbb{1})\Phi(X) \arrow[ul, "\pi_{\Phi(\mathbb{1}),\Phi(X)}"'] \arrow[r, "\Phi^0\mathrm{id}_{\Phi(X)}"] \arrow[r, phantom, shift left=4ex, "(2)"] \arrow[r, phantom, shift right=4ex, "(4)"] & A\Phi(X) \arrow[ur, "\pi_{A,\Phi(X)}"] \arrow[dr, "\lambda_{\Phi(X)}"] & (3) \\
\Phi(X) \arrow[uu, "\Phi^{\mathbb{1},X}"] \arrow[ur, "\widetilde{\Phi}^{\mathbb{1},X}"] \arrow[rrr, equal] & & & \Phi(X),
\end{tikzcd}$$

we get that (1) commutes by the definition of $\Phi^{\mathbb{1},X}$; (2) commutes by (2.21) and (3) commutes from the definition of $l^A_{\Phi(X)}$. The diagram (4) is the following:

$$\begin{tikzcd}
AAAXA \arrow[rr, "m_A\mathrm{id}_{AXA}"] & & AAXA \arrow[dd, "m_A\mathrm{id}_{XA}"] \\
AAXA \arrow[u, "\mathrm{id}_A\Delta_A\mathrm{id}_{XA}"] \arrow[r, "m_A\mathrm{id}_{XA}"] & AXA \arrow[ur, "\Delta_A\mathrm{id}_{XA}"] \arrow[dr, equal] & \\
AXA \arrow[u, "\mathrm{id}_A u_A\mathrm{id}_{XA}"] \arrow[ur, equal] & & AXA.
\end{tikzcd}$$

Here, the bottom left triangle commutes by the unit axiom of A, the top region commutes by the Frobenius compatibility condition between m_A and Δ_A, and the right triangle commutes by the assumption that A is special.

(j) Let us check that one of the Frobenius conditions holds for $\mathcal{C}$ strict; the rest is left to the reader. Consider the diagram below.

$$\begin{tikzcd}
\Phi(X)\otimes_A\Phi(X'X'') \arrow[rrrrr, "\Phi_{X,X'X''}"] \arrow[ddd, "\mathrm{id}_{\Phi(X)}\otimes_A\Phi^{X',X''}"'] & & & & & \Phi(XX'X'') \arrow[dddll, "\widetilde{\Phi}^{XX',X''}"'] \arrow[ddddd, "\Phi^{XX',X''}"] \\
 & & \Phi(X)\Phi(X'X'') \arrow[ull, "\pi_{\Phi(X),\Phi(X'X'')}"'] \arrow[urrr, "\widetilde{\Phi}_{X,X'X''}"] \arrow[d, "\mathrm{id}_{\Phi(X)}\Phi^{X'X''}"'] \arrow[dd, bend left, "\mathrm{id}_{\Phi(X)}\widetilde{\Phi}^{X'X''}"] \arrow[u, phantom, "(1)"] & & & \\
 & (2) & \Phi(X)(\Phi(X')\otimes_A\Phi(X'')) \arrow[dll, "\pi_{\Phi(X),\Phi(X')\otimes_A\Phi(X'')}"'] & (6) & (8) & \\
\Phi(X)\otimes_A(\Phi(X')\otimes_A\Phi(X'')) \arrow[dd, "{\alpha^A_{\Phi(X),\Phi(X'),\Phi(X'')}}^{-1}"] & (3) & \Phi(X)\Phi(X')\Phi(X'') \arrow[u, "\mathrm{id}_{\Phi(X)}\pi_{\Phi(X'),\Phi(X'')}"] \arrow[r, "\widetilde{\Phi}_{X,X'}\mathrm{id}_{\Phi(X'')}"] \arrow[d, "\pi_{\Phi(X),\Phi(X')}\mathrm{id}_{\Phi(X'')}"'] & \Phi(XX')\Phi(X'') \arrow[ddrr, "\pi_{\Phi(XX'),\Phi(X'')}"] & (7) & \\
 & & (\Phi(X)\otimes_A\Phi(X'))\Phi(X'') \arrow[ur, "\Phi_{X,X'}\mathrm{id}_{\Phi(X'')}"'] \arrow[dll, "\pi_{\Phi(X)\otimes_A\Phi(X'),\Phi(X'')}"] & (5) & (4) & \\
(\Phi(X)\otimes_A\Phi(X'))\otimes_A\Phi(X'') \arrow[rrrrr, "\Phi_{X,X'}\otimes_A\mathrm{id}_{\Phi(X'')}"'] & & & & & \Phi(XX')\otimes_A\Phi(X''),
\end{tikzcd}$$

The diagrams (2) and (4) commute from (2.21), and (3) commutes from the definition of the associativity constraint α^A. Moreover, (1) and (5) commute from the definition of $\Phi_{*,*}$, and (6) and (7) commute from the definition of $\Phi^{*,*}$. Lastly, (8) is the following diagram:

$$\begin{tikzcd}
AXAAX'X''A \arrow[r, "\mathrm{id}_{AX}m_A\mathrm{id}_{X'X''A}"] \arrow[d, "\mathrm{id}_{AXAAX'}u_A\mathrm{id}_{X''A}"] & AXAX'X''A \arrow[r, "\mathrm{id}_{AX}\varepsilon_A\mathrm{id}_{X'X''A}"] \arrow[d, "\mathrm{id}_{AXAX'}u_A\mathrm{id}_{X''A}"] & AXX'X''A \arrow[d, "\mathrm{id}_{AXX'}u_A\mathrm{id}_{X''A}"] \\
AXAAX'AX''A \arrow[r, "\mathrm{id}_{AX}m_A\mathrm{id}_{X'AX''A}"] \arrow[d, "\mathrm{id}_{AXAAX'}\Delta_A\mathrm{id}_{X''A}"] & AXAX'AX''A \arrow[r, "\mathrm{id}_{AX}\varepsilon_A\mathrm{id}_{X'AX''A}"] \arrow[d, "\mathrm{id}_{AXAX'}\Delta_A\mathrm{id}_{X''A}"] & AXX'AX''A \arrow[d, "\mathrm{id}_{AXX'}\Delta_A\mathrm{id}_{X''A}"] \\
AXAAX'AAX''A \arrow[r, "\mathrm{id}_{AX}m_A\mathrm{id}_{X'AAX''A}"] & AXAX'AAX''A \arrow[r, "\mathrm{id}_{AX}\varepsilon_A\mathrm{id}_{X'AAX''A}"] & AXX'AAX''A
\end{tikzcd}$$

where each square commutes because the maps are applied in different slots. $\square$

Remark 3.3. In the theorem above we gave the 'free' functor $\Phi : \mathcal{C} \to {}_A\mathcal{C}_A$ the structure of a Frobenius monoidal functor when the ground algebra A is Frobenius.

(a) Observe that Φ is not strong monoidal if $A \ncong \mathbb{1}_{\mathcal{C}}$.

(b) In the proof above, we did not need the full requirement that A is special; we only used the condition that $m_A\Delta_A = \mathrm{id}_A$.

(c) It is natural to consider connections to its (left or right) adjoint, the forgetful functor $U : {}_A\mathcal{C}_A \to \mathcal{C}$. It is discussed when U admits a Frobenius monoidal structure in [4, Theorem 6.2]; see also [33, Lemma 6.4].

In fact, we will employ the forgetful functor U in the next section to study the Morita equivalence of algebras in ${}_A\mathcal{C}_A$.

4. Morita equivalence of algebras in a category of bimodules

In this section, take $\mathcal{C}$ to be a rigid monoidal category and take A a special Frobenius algebra in $\mathcal{C}$. Our main result is on the Morita equivalence of algebras in the monoidal category of bimodules ${}_A\mathcal{C}_A$, given in Theorem 4.9 below. To begin, consider the following result and terminology.

Theorem 4.1. *Let $(\mathcal{S}, \otimes_{\mathcal{S}})$ and $(\mathcal{T}, \otimes_{\mathcal{T}})$ be monoidal categories. Take a monoidal functor $\Gamma : \mathcal{S} \to \mathcal{T}$ that preserves epimorphisms and so that the natural transformation $\Gamma_{*,*}$ of Γ is an epimorphism. If S and S' are Morita equivalent algebras in $\mathcal{S}$, then $\Gamma(S)$ and $\Gamma(S')$ are Morita equivalent algebras in $\mathcal{T}$.*

Proof. By Proposition 2.25(a), we have bimodules

$$\begin{aligned} &(\overline{P}, \quad \lambda^{S}_{\overline{P}} : S \otimes_{\mathcal{S}} \overline{P} \to \overline{P}, \quad \rho^{S'}_{\overline{P}} : \overline{P} \otimes_{\mathcal{S}} S' \to \overline{P}) \quad \in {}_S\mathcal{S}_{S'}, \\ &(\overline{Q}, \quad \lambda^{S'}_{\overline{Q}} : S' \otimes_{\mathcal{S}} \overline{Q} \to \overline{Q}, \quad \rho^{S}_{\overline{Q}} : \overline{Q} \otimes_{\mathcal{S}} S \to \overline{Q}) \quad \in {}_{S'}\mathcal{S}_{S}, \end{aligned}$$

equipped with isomorphisms $\overline{\tau} : \overline{P} \otimes_{S'} \overline{Q} \xrightarrow{\sim} S$ in ${}_S\mathcal{S}_S$ and $\overline{\mu} : \overline{Q} \otimes_S \overline{P} \xrightarrow{\sim} S'$ in ${}_{S'}\mathcal{S}_{S'}$. Take

$$P := \Gamma(\overline{P}), \qquad Q := \Gamma(\overline{Q}).$$

By Proposition A.1, we obtain the bimodules $(P, \lambda_P^{\Gamma(S)},\ \rho_P^{\Gamma(S')}) \in {}_{\Gamma(S)}\mathcal{T}_{\Gamma(S')}$ and $(Q, \lambda_Q^{\Gamma(S')},\ \rho_Q^{\Gamma(S)}) \in {}_{\Gamma(S')}\mathcal{T}_{\Gamma(S)}$, where

$$\begin{aligned} &\lambda_P^{\Gamma(S)} = \Gamma(\lambda^{S}_{\overline{P}})\, \Gamma_{S,\overline{P}} : \Gamma(S) \otimes_{\mathcal{T}} P \to P, \qquad &\rho_P^{\Gamma(S')} = \Gamma(\rho^{S'}_{\overline{P}})\, \Gamma_{\overline{P},S'} : P \otimes_{\mathcal{T}} \Gamma(S') \to P, \\ &\lambda_Q^{\Gamma(S')} = \Gamma(\lambda^{S'}_{\overline{Q}})\, \Gamma_{S',\overline{Q}} : \Gamma(S') \otimes_{\mathcal{T}} Q \to Q, \quad &\rho_Q^{\Gamma(S)} = \Gamma(\rho^{S}_{\overline{Q}})\, \Gamma_{\overline{Q},S} : Q \otimes_{\mathcal{T}} \Gamma(S) \to Q. \end{aligned}$$

Consider the morphisms, where $\otimes := \otimes_{\mathcal{S}}$ below:

$$\begin{aligned} \widehat{\tau} &:= \Gamma(\overline{\tau})\, \Gamma(\pi^{S'}_{\overline{P},\overline{Q}})\, \Gamma_{\overline{P},\overline{Q}} : P \otimes_{\mathcal{T}} Q \to \Gamma(\overline{P} \otimes \overline{Q}) \to \Gamma(\overline{P} \otimes_{S'} \overline{Q}) \to \Gamma(S), \\ \widehat{\mu} &:= \Gamma(\overline{\mu})\, \Gamma(\pi^{S}_{\overline{Q},\overline{P}})\, \Gamma_{\overline{Q},\overline{P}} : Q \otimes_{\mathcal{T}} P \to \Gamma(\overline{Q} \otimes \overline{P}) \to \Gamma(\overline{Q} \otimes_{S} \overline{P}) \to \Gamma(S'). \end{aligned}$$

Both $\widehat{\tau}$ and $\widehat{\mu}$ are epimorphisms (in ${}_{\Gamma(S)}\mathcal{T}_{\Gamma(S)}$ and ${}_{\Gamma(S')}\mathcal{T}_{\Gamma(S')}$, respectively) because the morphisms $\overline{\tau}$, $\overline{\mu}$, $\pi_{*,*}$ are each epic, the natural transformation $\Gamma_{*,*}$ of Γ is an epimorphism, and Γ preserves epimorphisms by assumption. Moreover, the epimorphisms $\widehat{\tau}$ and $\widehat{\mu}$ factor through epimorphisms

$$\begin{aligned} \tau : P \otimes_{\Gamma(S')} Q \twoheadrightarrow \Gamma(S) \quad &\in {}_{\Gamma(S)}\mathcal{T}_{\Gamma(S)}, \\ \mu : Q \otimes_{\Gamma(S)} P \twoheadrightarrow \Gamma(S') \quad &\in {}_{\Gamma(S')}\mathcal{T}_{\Gamma(S')}, \end{aligned}$$

so that

$$\widehat{\tau} = \tau\, \pi^{\Gamma(S')}_{P,Q}, \qquad \widehat{\mu} = \mu\, \pi^{\Gamma(S)}_{Q,P}. \tag{4.2}$$

Indeed,

$$\widehat{\tau}(\rho_P^{\Gamma(S')} \otimes \mathrm{id}_Q) = \widehat{\tau}(\mathrm{id}_P \otimes \lambda_Q^{\Gamma(S')})\alpha_{P,\Gamma(S'),Q},$$

which is verified by the commutative diagram below in the strict case. The regions commute due to the monoidal structure of Γ and by the definitions of $\rho_P^{\Gamma(S')}$, of $\lambda^{S'}_{\overline{Q}}$, of $\widehat{\tau}$, and of $P \otimes_{\Gamma(S')} Q$. Here, $\otimes := \otimes_{\mathcal{S}}$ in the diagram below.

$$
\begin{tikzcd}
P \otimes_{\mathcal{T}} \Gamma(S') \otimes_{\mathcal{T}} Q \arrow[r, "\mathrm{id}_P \otimes_{\mathcal{T}} \Gamma_{S',\overline{Q}}"'] \arrow[d, "\Gamma_{\overline{P},S'} \otimes_{\mathcal{T}} \mathrm{id}_Q"] \arrow[ddd, bend right=50, "\rho_P^{\Gamma(S')} \otimes_{\mathcal{T}} \mathrm{id}_Q"'] \arrow[rrr, bend left=15, two heads, "\mathrm{id}_P \otimes_{\mathcal{T}} \lambda_Q^{\Gamma(S')}"] & P \otimes_{\mathcal{T}} \Gamma(S' \otimes \overline{Q}) \arrow[rr, two heads, "\mathrm{id}_P \otimes_{\mathcal{T}} \Gamma(\lambda_{\overline{Q}}^{S'})"'] \arrow[d, "\Gamma_{\overline{P},S' \otimes \overline{Q}}"] & & P \otimes_{\mathcal{T}} Q \arrow[d, "\Gamma_{\overline{P},\overline{Q}}"'] \arrow[ddd, bend left=50, "\widehat{\tau}"] \\
\Gamma(\overline{P} \otimes S') \otimes_{\mathcal{T}} Q \arrow[r, "\Gamma_{\overline{P} \otimes S',\overline{Q}}"] \arrow[dd, "\Gamma(\rho_{\overline{P}}^{S'}) \otimes_{\mathcal{T}} \mathrm{id}_Q"] & \Gamma(\overline{P} \otimes S' \otimes \overline{Q}) \arrow[rr, "\Gamma(\mathrm{id}_{\overline{P}} \otimes \lambda_{\overline{Q}}^{S'})"] \arrow[dd, "\Gamma(\rho_{\overline{P}}^{S'} \otimes \mathrm{id}_{\overline{Q}})"] & & \Gamma(\overline{P} \otimes \overline{Q}) \arrow[d, "\Gamma(\pi_{\overline{P},\overline{Q}}^{S'})"'] \\
& & & \Gamma(\overline{P} \otimes_{S'} \overline{Q}) \arrow[d, "\Gamma(\overline{\tau})"'] \arrow[dl, equal] \\
P \otimes_{\mathcal{T}} Q \arrow[r, "\Gamma_{\overline{P},\overline{Q}}"] \arrow[rrr, bend right=10, two heads, "\widehat{\tau}"'] & \Gamma(\overline{P} \otimes \overline{Q}) \arrow[r, "\Gamma(\pi_{\overline{P},\overline{Q}}^{S'})"] & \Gamma(\overline{P} \otimes_{S'} \overline{Q}) \arrow[r, two heads, "\Gamma(\overline{\tau})"] & \Gamma(S)
\end{tikzcd}
$$

So the epimorphism τ exists by Definition 2.19. Likewise, the epimorphism μ exists. Finally, τ and μ satisfy diagrams $(*)$ and $(**)$ in Proposition 2.25(b) by Proposition A.2. Therefore, by Proposition 2.25(b), the algebras $\Gamma(S)$ and $\Gamma(S')$ are Morita equivalent in $\mathcal{T}$. $\square$

Definition 4.3. We call a monoidal functor $\Gamma : \mathcal{S} \to \mathcal{T}$ *Morita preserving* if it satisfies the conclusion of Theorem 4.1.

For the rest of the section, let $(\mathcal{C}, \otimes)$ be a rigid monoidal category, and recall that $\mathcal{C}$ is assumed to be abelian and locally small. Consider the following notation.

Notation 4.4 ($E := E(A)$)**.** Take $A \in \mathsf{Alg}(\mathcal{C})$ and denote by $E := E(A)$ the internal End object that represents the functor

$$
{}_A\mathcal{C}_A \to \mathsf{Set}, \quad X \mapsto \mathsf{Hom}_{\mathcal{C}_A}(A \otimes_A X, A);
$$

see [8, Section 7.9]. Namely, we take $M_1 = M_2 = A$ in [8, (7.20)].

Proposition 4.5. *If A is a Frobenius algebra in $\mathcal{C}$, then $E(A)$ is a Frobenius algebra in ${}_A\mathcal{C}_A$. If, further, A is special, then $E(A)$ also admits the structure of a special Frobenius algebra.*

Proof. The object structure of $E(A)$ follows from [8, Example 7.12.8] and the references within. In particular, $E(A) = {}^*A \otimes A$ with A-bimodule structure

$$
\begin{aligned}
\lambda_E^A = \quad & (r_{^*A} \otimes \mathrm{id}_A)(\mathrm{id}_{^*A} \otimes \mathsf{ev}'_A \otimes \mathrm{id}_A)(\alpha_{^*A,A,A} \otimes \mathrm{id}_A)(\mathrm{id}_{^*A} \otimes m_A \otimes \mathrm{id}_{^*AA}) \\
& \circ(\alpha_{^*A,A,A} \otimes \mathrm{id}_{^*AA})(\mathsf{coev}'_A \otimes \mathrm{id}_{A^*AA})(l_A^{-1} \otimes \mathrm{id}_{^*AA})\alpha_{A,^*A,A}^{-1}
\end{aligned}
$$

and $\rho_E^A = (\mathrm{id}_{^*A} \otimes m_A)\alpha_{^*A,A,A}$.

On the other hand, consider the Frobenius algebra $\mathbb{1}$ in $\mathcal{C}$. By Theorem 3.2, we then get that $\Phi(\mathbb{1}) \in \mathsf{FrobAlg}({}_A\mathcal{C}_A)$. Now by Remark 2.16, we have an isomorphism $\xi : {}^*A \xrightarrow{\sim} A$ in ${}_A\mathcal{C}$. So, we define a map

$$
\chi := \xi^{-1} r_A \otimes \mathrm{id}_A : \Phi(\mathbb{1}) = (A \otimes \mathbb{1}) \otimes A \longrightarrow {}^*A \otimes A = E.
$$

It is straight-forward to check that χ is an isomorphism of objects in ${}_A\mathcal{C}_A$. Since $\Phi(\mathbb{1})$ is Frobenius, E also admits the structure of a Frobenius algebra in ${}_A\mathcal{C}_A$.

Now suppose that A is special. Then $\varepsilon_A u_A = \varphi \operatorname{id}_{\mathbb{1}}$ for some nonzero $\varphi \in \mathbb{k}$. So we get that $m_{\Phi(\mathbb{1})}\, \Delta_{\Phi(\mathbb{1})} = \varphi \operatorname{id}_{\Phi(\mathbb{1})}$ in this case: indeed, by Proposition 2.9(a,b), we have

$$\begin{aligned} m_{\Phi(\mathbb{1})}\, \Delta_{\Phi(\mathbb{1})} &= \Phi(m_{\mathbb{1}})\; \Phi_{\mathbb{1},\mathbb{1}}\; \Phi^{\mathbb{1},\mathbb{1}}\; \Phi(\Delta_{\mathbb{1}}) \\ &= (\operatorname{id}_A \otimes m_{\mathbb{1}} \otimes \operatorname{id}_A)(\alpha_{A,\mathbb{1},\mathbb{1}} \otimes \operatorname{id}_A) \\ &\quad \circ (\operatorname{id}_{A\mathbb{1}} \otimes l_{\mathbb{1}} \otimes \operatorname{id}_A)(\operatorname{id}_{A\mathbb{1}} \otimes \varepsilon_A m_A \Delta_A u_A \otimes \operatorname{id}_{\mathbb{1}A})(\operatorname{id}_{A\mathbb{1}} \otimes l_{\mathbb{1}}^{-1} \otimes \operatorname{id}_A) \\ &\quad \circ (\alpha_{A,\mathbb{1},\mathbb{1}}^{-1} \otimes \operatorname{id}_A)(\operatorname{id}_A \otimes \Delta_{\mathbb{1}} \otimes \operatorname{id}_A) \\ &= \varphi \operatorname{id}_{\Phi(\mathbb{1})}, \end{aligned}$$

and

$$\begin{aligned} \varepsilon_{\Phi(\mathbb{1})}\; u_{\Phi(\mathbb{1})} &= \Phi^0\; \Phi(\varepsilon_{\mathbb{1}})\; \Phi(u_{\mathbb{1}})\; \Phi_0 \\ &= m_A(r_A \otimes \operatorname{id}_A)(\operatorname{id}_A \otimes \varepsilon_{\mathbb{1}} u_{\mathbb{1}} \otimes \operatorname{id}_A)(r_A^{-1} \otimes \operatorname{id}_A)\Delta_A \\ &= \operatorname{id}_A. \end{aligned}$$

By the isomorphism χ above, and one can then rescale the multiplication of E to yield that E is special. $\square$

By the proposition above, $E(A)$ is a special Frobenius algebra in ${}_A\mathcal{C}_A$, when A is special Frobenius. Now recall the functor

$$\Phi = \Phi_A^{\mathcal{C}} : \mathcal{C} \to {}_A\mathcal{C}_A$$

from Theorem 3.2, and consider the following functors:

$$\begin{aligned} \widehat{\Phi} := \Phi_E^{{}_A\mathcal{C}_A} :&\quad {}_A\mathcal{C}_A \to {}_E({}_A\mathcal{C}_A)_E, \\ \widehat{U} :&\quad {}_E({}_A\mathcal{C}_A)_E \to {}_A\mathcal{C}_A \qquad \text{(forget)}, \\ U :&\quad {}_A\mathcal{C}_A \to \mathcal{C} \qquad\qquad\quad \text{(forget)}. \end{aligned}$$

Corollary 4.6. *The functors Φ, $\widehat{\Phi}$, U, $\widehat{U}$ are each monoidal and Morita preserving.*

Proof. We have that Φ is monoidal by Theorem 3.2, and $\widehat{\Phi}$ is also monoidal by applying Theorem 3.2 with Proposition 4.5. Moreover, it is straight-forward to check that U is monoidal with the following structure: for $Y, Y' \in {}_A\mathcal{C}_A$, take

$$\begin{aligned} U_{Y,Y'} = \pi^A_{Y,Y'} :&\quad U(Y) \otimes U(Y') = Y \otimes Y' \to Y \otimes_A Y' = U(Y \otimes_A Y'), \\ U_0 = u_A :&\quad \mathbb{1} \to A = U(\mathbb{1}_{{}_A\mathcal{C}_A}). \end{aligned} \tag{4.7}$$

For instance, the following diagram commutes due to the unit constraint on Y (as a left A-module in $\mathcal{C}$) and by definition of l^A_Y:

$$\begin{tikzcd} \mathbb{1} \otimes U(Y) = \mathbb{1} \otimes Y \arrow[rr, "U_0 \otimes \operatorname{id}_Y = u_A \otimes \operatorname{id}_Y"] \arrow[d, "l_Y"'] & & A \otimes Y = U(\mathbb{1}_{{}_A\mathcal{C}_A}) \otimes U(Y) \arrow[d, "U_{A,Y} = \pi^A_{A,Y}"] \arrow[dll, "\lambda_Y"'] \\ U(Y) = Y & & A \otimes_A Y = U(A \otimes_A Y) \arrow[ll, "l^A_Y"'] \end{tikzcd}$$

In a similar manner, the functor $\widehat{U}$ has a monoidal structure.

Next, we apply Theorem 4.1 to get that each of Φ, $\widehat{\Phi}$, U, $\widehat{U}$ are Morita preserving. Indeed, it is clear from Theorem 3.2 that the natural transformations $\Phi_{*,*}$ and $\widehat{\Phi}_{*,*}$ are epimorphisms. Moreover, Φ and $\widehat{\Phi}$ are left adjoints (to U and $\widehat{U}$, respectively), so they preserve epimorphisms. On the other hand, we see that the natural transformations $U_{*,*}$ and $\widehat{U}_{*,*}$ are epimorphisms from (4.7). Lastly, U and $\widehat{U}$ preserve epimorphisms as they are faithful functors. □

Now we establish the main result of the section. But first we need a preliminary result on the algebra $U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B)$ in $\mathcal{C}$ (resulting from the corollary above).

Lemma 4.8. *For A a special Frobenius algebra in $\mathcal{C}$ and $B \in \mathsf{Alg}(\mathcal{C})$, we get the following statements.*

(a) *$D := U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B) \in \mathsf{Alg}(\mathcal{C})$. Here, $D = (E \otimes_A ((A \otimes B) \otimes A)) \otimes_A E$ as an object in $\mathcal{C}$.*

(b) *D is isomorphic to*

$$T := (U(E) \otimes B) \otimes U(E) = (E \otimes B) \otimes E$$

as objects in $\mathcal{C}$ via

$$\theta := (\nu_E^A \otimes \mathrm{id}_B \otimes \mu_E^A)(\overline{\alpha}^{-1}_{E,A,B} \otimes \mathrm{id}_{AE})\,\overline{\alpha}_{EAB,A,E}\,(\overline{\alpha}^{-1}_{E,AB,A} \otimes_A \mathrm{id}_E) : D \xrightarrow{\sim} T,$$

for natural isomorphisms $\mu_E^A : A \otimes_A E \xrightarrow{\sim} E$ and $\nu_E^A : E \otimes_A A \xrightarrow{\sim} E$, and associativity constraint $\overline{\alpha}$ given in Lemma 2.24.

(c) *T admits the structure of an algebra in $\mathcal{C}$, with $m_T = \theta\, m_D\, (\theta^{-1} \otimes \theta^{-1})$ and $u_T = \theta\, u_D$, and $D \cong T$ as algebras in $\mathcal{C}$.*

Proof. Part (a) follows from Corollary 4.6 and Proposition 2.9. Part (b) holds because μ_E^A and ν_E^A are isomorphisms in $\mathcal{C}$ (see [8, Exercise 7.8.22]), and $\overline{\alpha}$ is an isomorphism in $\mathcal{C}$ by Lemma 2.24. Part (c) follows from parts (a) and (b). □

Theorem 4.9. *Take A a special Frobenius algebra in $\mathcal{C}$, and take $B, B' \in \mathsf{Alg}(\mathcal{C})$. Then, B and B' are Morita equivalent as algebras in $\mathcal{C}$ if and only if $\Phi(B)$ and $\Phi(B')$ are Morita equivalent as algebras in ${}_A\mathcal{C}_A$.*

Proof. The forward direction holds because Φ is Morita preserving by Corollary 4.6.

For the converse, note that the algebras $U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B)$ and $U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B')$ are Morita equivalent algebras in $\mathcal{C}$ because $U, \widehat{U}, \widehat{\Phi}$ are each Morita preserving [Corollary 4.6]. So it suffices to show that $U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B)$ is Morita equivalent to B as algebras in $\mathcal{C}$, which we achieve as follows.

By Lemma 4.8, $D := U\,\widehat{U}\,\widehat{\Phi}\,\Phi(B)$ is isomorphic to $T = (U(E) \otimes B) \otimes U(E) = (E \otimes B) \otimes E$ as algebras in $\mathcal{C}$. So it suffices to show that T is Morita equivalent to B in $\mathcal{C}$. This holds using the methods in the proof of Proposition 2.26(b). We discuss this in the strict case and leave the general case to the reader.

We have by Remark 2.8 and Proposition 4.5 that E is a self-dual object in ${}_A\mathcal{C}_A$ with evaluation map $p_E = \varepsilon_E m_E : E \otimes_A E \to A$. To proceed, recall Notation 3.1 and define the morphism

$$\phi := \varepsilon_A\, \widetilde{p}_E : E \otimes E \longrightarrow \mathbb{1}.$$

Now, take $P = E \otimes B$ with morphisms $\lambda_P^T = (\mathrm{id}_E \otimes m_B)(\mathrm{id}_{EB} \otimes \phi \otimes \mathrm{id}_B)$ and $\rho_P^B = (\mathrm{id}_E \otimes m_B)$, and take $Q = B \otimes E$ with morphisms $\lambda_Q^B = (m_B \otimes \mathrm{id}_E)$ and $\rho_Q^T = (m_B \otimes \mathrm{id}_E)(\mathrm{id}_B \otimes \phi \otimes \mathrm{id}_{BE})$. We then obtain that $P \in {}_T\mathcal{C}_B$ and $Q \in {}_B\mathcal{C}_T$. Moreover, we have epimorphisms

$$\begin{aligned} \widehat{\tau} = &\quad \mathrm{id}_E \otimes m_B \otimes \mathrm{id}_E &: P \otimes Q \to T &\quad \in {}_T\mathcal{C}_T, \\ \widehat{\mu} = &\quad m_B(\mathrm{id}_B \otimes \phi \otimes \mathrm{id}_B) &: Q \otimes P \to B &\quad \in {}_B\mathcal{C}_B, \end{aligned}$$

which factor through epimorphisms $\tau : P \otimes_B Q \to T$ and $\mu : Q \otimes_T P \to B$, respectively. Similar to the proof of Proposition 2.26(b), it is also straight-forward to check that τ and μ satisfy diagrams $(*)$ and $(**)$ of Proposition 2.25(b). Thus, by Proposition 2.25, T and B are Morita equivalent in $\mathcal{C}$, as desired. $\square$

5. Algebras $A(L, \psi)$ in pointed fusion categories Vec_G^ω

We recall here the definition of a twisted group algebra in the pointed fusion category Vec_G^ω [Definition 5.5]. We show that these algebras can be given the structure of a Frobenius algebra in Vec_G^ω [Proposition 5.7], and further, that they enjoy nice properties [Proposition 5.9]. Let us begin with discussing pointed tensor categories.

Definition 5.1. Let $\mathcal{C}$ be a tensor category. It is called *pointed* if all of its simple objects are invertible, in the sense that the co/evaluation maps on simple objects are isomorphisms in $\mathcal{C}$.

The following pointed fusion category will be crucial to our work.

Definition 5.2 (Vec_G^ω, δ_g)**.** Take G a finite group with 3-cocycle $\omega \in H^3(G, \Bbbk^\times)$. The category Vec_G^ω is the category of G-graded vector spaces $V = \bigoplus_{x \in G} V_x$ with associativity constraint ω given as follows. In particular, its simple objects are $\{\delta_g\}_{g \in G}$, where the G-grading is $(\delta_g)_x = \delta_{g,x} \cdot \Bbbk$, for $g, x \in G$. Morphisms are $\Bbbk$-linear maps that preserve the G-grading.

The monoidal structure is determined by the G-grading of objects

$$(V \otimes W)_x = \bigoplus_{yz = x} V_y \otimes W_z,$$

the associativity constraint

$$\alpha_{\delta_g, \delta_{g'}, \delta_{g''}} = \omega^{-1}(g, g', g'')\, \mathrm{id}_{\delta_{gg'g''}} : (\delta_g \otimes \delta_{g'}) \otimes \delta_{g''} \to \delta_g \otimes (\delta_{g'} \otimes \delta_{g''}),$$

the unit object δ_e, with unit constraints $l_{\delta_g} = \omega^{-1}(e, e, g)\, \mathrm{id}_{\delta_g}$, $r_{\delta_g} = \omega(g, e, e)\, \mathrm{id}_{\delta_g}$.

The duals of simple objects are defined as $\delta_g^* = \delta_{g^{-1}} = {}^*\delta_g$, with evaluation morphisms given by $\mathsf{ev}_{\delta_g}(\delta_g^* \otimes \delta_g) = \omega(g, g^{-1}, g)\delta_e$ and $\mathsf{ev}'_{\delta_g}(\delta_g \otimes {}^*\delta_g) = \omega^{-1}(g, g^{-1}, g)\delta_e$, and coevaluation morphisms given by $\mathsf{coev}_{\delta_g}(\delta_e) = \delta_g \otimes \delta_g^*$ and $\mathsf{coev}'_{\delta_g}(\delta_e) = {}^*\delta_g \otimes \delta_g$.

Remark 5.3. (a) We assume as in [8, Remark 2.6.3] that ω is normalized; in particular, $l_{\delta_g} = \mathrm{id}_{\delta_g} = r_{\delta_g}$.

(b) The associativity constraint $\alpha_{\delta_g,\delta_{g'},\delta_{g''}}$ of Vec_G^ω given in [8] is defined by $\omega(g,g',g'')\mathrm{id}_{\delta_{gg'g''}}$, but we need to use $\omega^{-1}(g,g',g'')\ \mathrm{id}_{\delta_{gg'g''}}$ here in order to get that the twisted group algebra $A(L,\psi)$ presented in Definition 5.5 below is an associative algebra in Vec_G^ω.

Not only is Vec_G^ω a pointed fusion category, we have that every pointed fusion category is equivalent to one of this type (see [10, Section 8.8]).

For reference in computations later, the 3-cocycle condition on ω is

$$\omega(g_1g_2,g_3,g_4)\,\omega(g_1,g_2,g_3g_4) \;=\; \omega(g_1,g_2,g_3)\,\omega(g_1,g_2g_3,g_4)\,\omega(g_2,g_3,g_4) \tag{5.4}$$

for all $g_i \in G$.

Next we turn our attention to algebras in, and module categories over, Vec_G^ω. To continue, consider the following terminology.

Definition 5.5 ($A(L,\psi)$)**.** Take L a subgroup of G so that $\omega|_{L^{\times 3}}$ is trivial, and take $\psi \in C^2(L,\Bbbk^\times)$ so that $d\psi = \omega|_{L^{\times 3}}$. We assume that ψ is normalized. We define the *twisted group algebra* $A(L,\psi)$ in Vec_G^ω to be $\bigoplus_{g\in L}\delta_g$ as an object in Vec_G^ω, with multiplication given by

$$\delta_g \otimes \delta_{g'} \mapsto \psi(g,g')\delta_{gg'}.$$

It is well-known, and we will see later in Proposition 5.7, that $A(L,\psi)$ is indeed an associative algebra in Vec_G^ω.

For reference in computations later, note that for a 2-cocycle, say θ, on a subgroup N of G the condition that $d\theta = \omega|_{N^{\times 3}}$ is translated as follows:

$$\theta(f_1,f_2f_3)\,\theta(f_2,f_3) \;=\; \omega(f_1,f_2,f_3)\,\theta(f_1f_2,f_3)\,\theta(f_1,f_2) \tag{5.6}$$

for $f_i \in N$.

We show now that twisted group algebras $A(L,\psi)$ are Frobenius algebras in Vec_G^ω.

Proposition 5.7. *The twisted group algebra $A(L,\psi)$ admits the structure of a Frobenius algebra in Vec_G^ω: for $g,g' \in L$, it is given by*

$$\begin{aligned}
m_{A(L,\psi)}(\delta_g\otimes\delta_{g'}) &= \psi(g,g')\ \delta_{gg'},\\
u_{A(L,\psi)}(\delta_e) &= \delta_e,\\
\Delta_{A(L,\psi)}(\delta_g) &= |L|^{-1}\bigoplus_{h\in L}\psi^{-1}(gh,h^{-1})\ [\delta_{gh}\otimes\delta_{h^{-1}}],\\
\varepsilon_{A(L,\psi)}(\delta_g) &= \delta_{g,e}\ |L|\ \delta_e.
\end{aligned}$$

Proof. We start by showing that $A(L,\psi)$ is an algebra in Vec_G^ω. For the associativity of multiplication, consider the following calculation:

$$\begin{aligned}
&m_{A(L,\psi)}(\mathrm{id}\otimes m_{A(L,\psi)})\ \alpha[(\delta_g\otimes\delta_{g'})\otimes\delta_{g''}]\\
&= \omega^{-1}(g,g',g'')\ m_{A(L,\psi)}(\mathrm{id}\otimes m_{A(L,\psi)})[\delta_g\otimes(\delta_{g'}\otimes\delta_{g''})]\\
&= \omega^{-1}(g,g',g'')\ \psi(g',g'')\ m_{A(L,\psi)}\,(\delta_g\otimes\delta_{g'g''})\\
&= \omega^{-1}(g,g',g'')\ \psi(g',g'')\ \psi(g,g'g'')\ \delta_{gg'g''}\\
&= \psi(g,g')\ \psi(gg',g'')\ \delta_{gg'g''}
\end{aligned}$$

$$
\begin{aligned}
&= \psi(g,g')\, m_{A(L,\psi)}\,(\delta_{gg'} \otimes \delta_{g''}) \\
&= m_{A(L,\psi)}(m_{A(L,\psi)} \otimes \mathrm{id})[(\delta_g \otimes \delta_{g'}) \otimes \delta_{g''}].
\end{aligned}
$$

For the fourth equation, we used (5.6) with $\theta = \psi$ and $(f_1, f_2, f_3) = (g, g', g'')$. Next, for $u_{A(L,\psi)}$ to satisfy the unit constraint, recall that ψ is normalized, and we get

$$
\begin{aligned}
m_{A(L,\psi)}(u_{A(L,\psi)} \otimes \mathrm{id})(\delta_e \otimes \delta_g) \;&=\; m_{A(L,\psi)}(\delta_e \otimes \delta_g) \;=\; \psi(e,g)\delta_g \;=\; \delta_g, \\
m_{A(L,\psi)}(\mathrm{id} \otimes u_{A(L,\psi)})(\delta_g \otimes \delta_e) \;&=\; m_{A(L,\psi)}(\delta_g \otimes \delta_e) \;=\; \psi(g,e)\delta_g \;=\; \delta_g.
\end{aligned}
$$

Thus, $A(L,\psi)$ is an algebra in Vec_G^ω.

Next we define a nondegenerate pairing p, with copairing q, on $A(L,\psi)$. Take

$$
p\,(\delta_g \otimes \delta_{g'}) := \begin{cases} |L|\ \psi(g,g')\ \delta_e, & gg' = e \\ 0, & gg' \neq e \end{cases}
$$

and

$$
q(\delta_e) := |L|^{-1} \bigoplus_{h \in L} \psi^{-1}(h,h^{-1})\ [\delta_h \otimes \delta_{h^{-1}}]. \tag{5.8}
$$

Note that

$$
\begin{aligned}
p\ (\mathrm{id} \otimes m)\ \alpha\ [(\delta_g \otimes \delta_{g'}) \otimes \delta_{g''}] &= \omega^{-1}(g,g',g'')\ p\ (\mathrm{id} \otimes m)[\delta_g \otimes (\delta_{g'} \otimes \delta_{g''})] \\
&= \omega^{-1}(g,g',g'')\ \psi(g',g'')\ p\ [\delta_g \otimes \delta_{g'g''}] \\
&= |L|\ \omega^{-1}(g,g',g'')\ \psi(g',g'')\ \psi(g,g'g'')\delta_{gg'g'',e}\ \delta_e \\
&= |L|\ \psi(g,g')\ \psi(gg',g'')\delta_{gg'g'',e}\ \delta_e \\
&= \psi(g,g')\ p\ [\delta_{gg'} \otimes \delta_{g''}] \\
&= p\ (m \otimes \mathrm{id})\ [(\delta_g \otimes \delta_{g'}) \otimes \delta_{g''}],
\end{aligned}
$$

where the fourth equality holds by (5.6) with $\theta = \psi$ and $(f_1, f_2, f_3) = (g, g', g'')$. So, it suffices to verify the Snake Equation for p and q:

$$
r_A\ (\mathrm{id}_A \otimes p)\ \alpha_{A,A,A}\ (q \otimes \mathrm{id}_A)\ l_A^{-1} \;=\; \mathrm{id}_A \;=\; l_A\ (p \otimes \mathrm{id}_A)\ \alpha^{-1}_{A,A,A}\ (\mathrm{id}_A \otimes q)\ r_A^{-1}.
$$

Now,

$$
\begin{aligned}
&[r_A\ (\mathrm{id}_A \otimes p)\ \alpha_{A,A,A}\ (q \otimes \mathrm{id}_A)\ l_A^{-1}](\delta_g) \\
&= |L|^{-1} \textstyle\bigoplus_{h \in L} \psi^{-1}(h,h^{-1})\ r_A\ (\mathrm{id}_A \otimes p)\ \alpha_{A,A,A}[(\delta_h \otimes \delta_{h^{-1}}) \otimes \delta_g] \\
&= |L|^{-1} \textstyle\bigoplus_{h \in L} \psi^{-1}(h,h^{-1})\ \omega^{-1}(h,h^{-1},g)\ r_A\ (\mathrm{id}_A \otimes p)[\delta_h \otimes (\delta_{h^{-1}} \otimes \delta_g)] \\
&= \psi^{-1}(g,g^{-1})\ \omega^{-1}(g,g^{-1},g)\ \psi(g^{-1},g)\ \delta_g \\
&= \delta_g,
\end{aligned}
$$

where the last equation holds by (5.6) for $(f_1, f_2, f_3) = (g, g^{-1}, g)$. On the other hand,

$$
\begin{aligned}
&[l_A\ (p \otimes \mathrm{id}_A)\ \alpha^{-1}_{A,A,A}\ (\mathrm{id}_A \otimes q)\ r_A^{-1}](\delta_g) \\
&= |L|^{-1} \textstyle\bigoplus_{h \in L} \psi^{-1}(h,h^{-1})\ l_A\ (p \otimes \mathrm{id}_A)\ \alpha^{-1}_{A,A,A}\ [\delta_g \otimes (\delta_h \otimes \delta_{h^{-1}})] \\
&= |L|^{-1} \textstyle\bigoplus_{h \in L} \psi^{-1}(h,h^{-1})\ \omega(g,h,h^{-1})\ l_A\ (p \otimes \mathrm{id}_A)[(\delta_g \otimes \delta_h) \otimes \delta_{h^{-1}})]
\end{aligned}
$$

$$= \psi^{-1}(g^{-1}, g)\, \omega(g, g^{-1}, g)\, \psi(g, g^{-1})\, \delta_g$$
$$= \delta_g,$$

where, again, the last equation holds by (5.6) for $(f_1, f_2, f_3) = (g, g^{-1}, g)$. Hence, by Remark 2.8, $A(L, \psi)$ is a Frobenius algebra in Vec_G^ω.

For the Frobenius algebra (A, m, u, p, q) above, the comultiplication map Δ and counit map ε are given as follows (again due to Remark 2.8):

$$\begin{aligned}\Delta(\delta_g) &= (m \otimes \mathrm{id}_A)\, \alpha^{-1}_{A,A,A}\, (\mathrm{id}_A \otimes q)\, r_A^{-1}(\delta_g)\\ &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(h, h^{-1})\, (m \otimes \mathrm{id}_A)\, \alpha^{-1}[\delta_g \otimes (\delta_h \otimes \delta_{h^{-1}})]\\ &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(h, h^{-1})\, \omega(g, h, h^{-1})\, (m \otimes \mathrm{id})[(\delta_g \otimes \delta_h) \otimes \delta_{h^{-1}}]\\ &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(h, h^{-1})\, \omega(g, h, h^{-1})\, \psi(g, h)\, [\delta_{gh} \otimes \delta_{h^{-1}}]\\ &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(gh, h^{-1})\, [\delta_{gh} \otimes \delta_{h^{-1}}].\end{aligned}$$

Here, the ultimate equality holds by applying (5.6) to $(f_1, f_2, f_3) = (g, h, h^{-1})$. Moreover,

$$\varepsilon(\delta_g) \;=\; p\, (u \otimes \mathrm{id}_A)\, r_A^{-1}(\delta_g) \;=\; p(\delta_g \otimes \delta_e) \;=\; \delta_{g,e}\, |L|\, \delta_e.$$

Therefore, $A(L, \psi) \in \mathsf{FrobAlg}(\mathsf{Vec}_G^\omega)$. $\square$

Now we discuss algebraic properties of twisted group algebras; see Section 2.2.

Proposition 5.9. *The twisted group algebra $A(L, \psi)$, with structural morphisms $m, u, \Delta, \varepsilon$ given in Proposition 5.7, possesses the following properties:*

(a) *connected;*
(b) *indecomposable;*
(c) *special;*
(d) *separable; and*
(e) *symmetric Frobenius if and only if $\omega(g^{-1}, g, g^{-1}) = 1$ for each $g \in L$.*

Proof. Denote $A := A(L, \psi)$.

(a) We have that $\mathrm{Hom}_{\mathsf{Vec}_G^\omega}(\mathbb{1}_{\mathsf{Vec}_G^\omega}, A) = \mathrm{Hom}_{\mathsf{Vec}_G^\omega}(\delta_e, \oplus_{g\in L}\delta_g) = \{\delta_e \mapsto \delta_e\}$, because morphisms preserve G-grading. Then $\dim \mathrm{Hom}_{\mathsf{Vec}_G^\omega}(\mathbb{1}_{\mathsf{Vec}_G^\omega}, A) = 1$ and A is connected.

(b) By way of contradiction, suppose that $A = A_1 \oplus A_2$ is a decomposable algebra. Then δ_g is a summand of A_1 for some $g \in L$. Since A_1 is closed under multiplication, we get that $m(\delta_g \otimes \delta_g) = \psi(g, g)\delta_{g^2}$ is a summand of A_1. Repeating this process, we obtain that $\psi(g, g)\psi(g^2, g) \cdots \psi(g^{n-1}, g)\delta_e$ is a summand of A_1, for $n = \mathrm{ord}(g)$. Since ψ takes values in $\Bbbk^\times$, we can rescale to obtain that δ_e is a summand of A_1. Likewise, δ_e is also a summand of A_2, which contradicts $A_1 \cap A_2 = (0)$. Therefore, A is an indecomposable algebra in Vec_G^ω.

(c) The algebra A is special because

$$\begin{aligned}m\Delta(\delta_g) &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(gh, h^{-1})\, m[(\delta_{gh} \otimes \delta_{h^{-1}})]\\ &= |L|^{-1} \textstyle\bigoplus_{h\in L} \psi^{-1}(gh, h^{-1})\, \psi(gh, h^{-1})\, \delta_g \quad = \delta_g,\end{aligned}$$

and $\varepsilon_{A(L,\psi)}\, u_{A(L,\psi)}(\delta_e) = |L|\delta_e = |L|\, \mathrm{id}_{\mathbb{1}}(\delta_e)$.

(d) This follows from Remark 2.11(b) and part (c) above.

(e) Since ${}^*\delta_g = \delta_{g^{-1}} = \delta_g^*$ for all $g \in L$, we have that ${}^*A = A^*$ as objects in Vec_G^ω. Moreover, recall Definition 2.12 and observe that

$$\varepsilon m(\delta_g \otimes \delta_{g'}) \;=\; \delta_{gg',e}\,|L|\,\psi(g,g')\,\delta_e,$$

and that $\Omega_A(\delta_g \otimes \delta_{g'})$, is equal to

$$\begin{aligned}
&[\mathsf{ev}_A(\mathrm{id}_{A^*}\otimes l_A)(\mathrm{id}_{A^*}\otimes \varepsilon m \otimes \mathrm{id}_A)(\alpha\otimes \mathrm{id}_A)((\mathsf{coev}'_A\otimes \mathrm{id}_A)l_A^{-1}\otimes \mathrm{id}_A)](\delta_g\otimes\delta_{g'})\\
&= \bigoplus_{h\in L}[\mathsf{ev}_A(\mathrm{id}_{A^*}\otimes l_A)(\mathrm{id}_{A^*}\otimes \varepsilon m\otimes \mathrm{id}_A)(\alpha\otimes\mathrm{id}_A)]((((\delta_{h^{-1}}\otimes\delta_h)\otimes\delta_g)\otimes\delta_{g'})\\
&= \bigoplus_{h\in L}\omega^{-1}(h^{-1},h,g)[\mathsf{ev}_A(\mathrm{id}_{A^*}\otimes l_A)(\mathrm{id}_{A^*}\otimes\varepsilon m\otimes\mathrm{id}_A)]((\delta_{h^{-1}}\otimes(\delta_h\otimes\delta_g))\otimes\delta_{g'})\\
&= |L|\bigoplus_{h\in L}\delta_{hg,e}\,\omega^{-1}(h^{-1},h,g)\,\psi(h,g)\,\mathsf{ev}_A(\delta_{h^{-1}}\otimes\delta_{g'})\\
&= |L|\,\omega^{-1}(g,g^{-1},g)\,\psi(g^{-1},g)\,\mathsf{ev}_A(\delta_g\otimes\delta_{g'})\\
&= \oplus_{t\in L}|L|\omega^{-1}(g,g^{-1},g)\,\psi(g^{-1},g)\,\mathsf{ev}_{\delta_t}(\delta_g\otimes\delta_{g'})\\
&= \delta_{gg',e}|L|\omega^{-1}(g,g^{-1},g)\,\psi(g^{-1},g)\,\mathsf{ev}_{\delta_{g^{-1}}}(\delta_{g^{-1}}^*\otimes\delta_{g^{-1}})\\
&= \delta_{gg',e}|L|\omega^{-1}(g,g^{-1},g)\,\psi(g^{-1},g)\,\omega(g^{-1},g,g^{-1})\delta_e.
\end{aligned}$$

Since A is Frobenius by Proposition 5.7, by the computations above we get that A is symmetric Frobenius if and only if if and only if

$$\psi(g,g^{-1}) = \omega^{-1}(g,g^{-1},g)\;\psi(g^{-1},g)\;\omega(g^{-1},g,g^{-1}).$$

From (5.6) with $f_1 = g, f_2 = g^{-1}, f_3 = g$, this is equivalent to $\omega(g^{-1},g,g^{-1}) = 1$ for all $g \in L$. $\square$

6. Algebras $A^{K,\beta}(L,\psi)$ in group-theoretical fusion categories $\mathcal{C}(G,\omega,K,\beta)$

We define in this section the main structures of interest in this work: twisted Hecke algebras [Definition 6.3]. These are algebras in group-theoretical fusion categories $\mathcal{C}$ [Definition 6.1] that are analogous to the twisted group algebras in Vec_G^ω discussed in Section 5. We establish that the twisted Hecke algebras admit the structure of a Frobenius algebra in $\mathcal{C}$ [Theorem 6.4], and further, as algebras in $\mathcal{C}$ we show that they are indecomposable, separable, and special [Proposition 6.9]. We also discuss when these (Frobenius) algebras are connected in $\mathcal{C}$ [Proposition 6.11]. We begin by introducing the terminology mentioned above.

Definition 6.1 ($\mathcal{C}(G,\omega,K,\beta)$)**.** [10, Section 8.8; Definition 8.40] A *group-theoretical fusion category* is a category of bimodules of the form

$$\mathcal{C}(G,\omega,K,\beta) := {}_{A(K,\beta)}(\mathsf{Vec}_G^\omega)_{A(K,\beta)}$$

for a twisted group algebra $A(K,\beta)$ in Vec_G^ω.

This is equivalent to the functor category $\mathsf{Fun}_{\mathsf{Vec}_G^\omega}(\mathcal{M}(K,\beta),\mathcal{M}(K,\beta))^{\mathrm{op}}$; see [8, Proposition 7.11.1, Definition 7.12.2, and Remark 7.12.5]. Next, we recall a description of simple objects of group-theoretical fusion categories.

Lemma 6.2. [8, Example 9.7.4] [20, Section 5] *Any simple object of* $\mathcal{C}(G,\omega,K,\beta)$ *is of the form*

$$V_{g,\rho} = \left(\bigoplus_{f\in K, k\in T} (\delta_f \otimes \delta_g) \otimes \delta_k \right)^{\oplus n_g},$$

where $g \in G$ *is a representative of a double coset in* $K\backslash G/K$, T *is a set of representatives of the classes in* $K/K^{g^{-1}}$ *for* $K^{g^{-1}} := (K \cap g^{-1}Kg)$, $\rho : K^{g^{-1}} \to \mathrm{GL}(V)$ *is a certain irreducible projective representation, and* $n_g = \dim V$. *The* $A(K,\beta)$*-bimodule structure on* $V_{g,\rho}$ *is given by the left* $A(K,\beta)$*-action* $m_{A(K,\beta)} \otimes \mathrm{id} \otimes \mathrm{id}$ *and the compatible right* $A(K,\beta)$*-action is determined by the left* $A(K,\beta)$*-action and* ρ. □

Now we turn our attention to algebraic structures in group-theoretical fusion categories.

Definition 6.3 ($A^{K,\beta}(L,\psi)$)**.** Consider the functor

$$\Phi : \mathsf{Vec}_G^{\omega} \to \mathcal{C}(G,\omega,K,\beta)$$

from Theorem 3.2 in the case when $\mathcal{C} = \mathsf{Vec}_G^{\omega}$ and $A = A(K,\beta)$. We refer to

$$\Phi(A(L,\psi)) =: A^{K,\beta}(L,\psi)$$

as a *twisted Hecke algebra* in $\mathcal{C}(G,\omega,K,\beta)$.

We use this terminology because the simple objects of the group-theoretical fusion category $\mathcal{C}(G,\omega,K,\beta)$ are in part parameterized by K-double cosets in G [Lemma 6.2], and as we see below, the multiplication of the algebra is twisted by cocycles.

Theorem 6.4. *The twisted Hecke algebra* $A^{K,\beta}(L,\psi)$ *equals*

$$\bigoplus_{g\in L;\ f,k\in K} (\delta_f \otimes \delta_g) \otimes \delta_k$$

as an object in $\mathcal{C}(G,\omega,K,\beta)$. *Furthermore, for* $f,f',k,k',d,d' \in K$ *and* $g,g' \in L$, *we have the following statements.*

(a) $A^{K,\beta}(L,\psi)$ *has the structure of an algebra in* $\mathcal{C}(G,\omega,K,\beta)$, *where*

$$\begin{aligned} &m_{A^{K,\beta}(L,\psi)}[((\delta_f \otimes \delta_g)\otimes \delta_k) \ \otimes_{A(K,\beta)} \ ((\delta_{f'} \otimes \delta_{g'})\otimes \delta_{k'})] \\ &\quad = \ \delta_{kf',e}\, \omega(fgk, f'g', k')\, \omega^{-1}(fg, k, f'g')\, \omega(k, f', g')\, \omega^{-1}(f, g, g') \\ &\quad\quad \cdot \ \beta(k, f')\, \psi(g, g')\, [(\delta_f \otimes \delta_{gg'}) \otimes \delta_{k'}], \end{aligned}$$

$$u_{A^{K,\beta}(L,\psi)}(\delta_d) \ = \ \textstyle\bigoplus_{s\in K} \beta^{-1}(ds^{-1}, s)\, [(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s].$$

(b) *With the above, $A^{K,\beta}(L,\psi)$ is a Frobenius algebra in $\mathcal{C}(G,\omega,K,\beta)$, where*

$$\begin{aligned}
&\Delta_{A^{K,\beta}(L,\psi)}[(\delta_f \otimes \delta_g) \otimes \delta_k] \\
&= |K|^{-1}|L|^{-1} \bigoplus\nolimits_{h\in L;\ s\in K}\ \omega(f,gh,h^{-1})\,\omega(fgh,s,s^{-1}h^{-1})\,\omega^{-1}(fghs,s^{-1}h^{-1},k) \\
&\qquad\qquad \cdot\ \omega^{-1}(s,s^{-1},h^{-1})\,\psi^{-1}(gh,h^{-1})\,\beta^{-1}(s,s^{-1}) \\
&\qquad\qquad \cdot\ [((\delta_f \otimes \delta_{gh}) \otimes \delta_s)\ \otimes_{A(K,\beta)}\ ((\delta_{s^{-1}} \otimes \delta_{h^{-1}}) \otimes \delta_k)],
\end{aligned}$$

$$\varepsilon_{A^{K,\beta}(L,\psi)}[(\delta_f \otimes \delta_g) \otimes \delta_k] \ = \ \delta_{g,e}\,|L|\,\beta(f,k)\,\delta_{fk}.$$

Proof. The definition of the functor Φ gives us

$$A^{K,\beta}(L,\psi) := \Phi(A(L,\psi)) = (A(K,\beta) \otimes A(L,\psi)) \otimes A(K,\beta),$$

which corresponds to the object

$$\bigoplus_{g\in L;\ f,k\in K} (\delta_f \otimes \delta_g) \otimes \delta_k$$

in the category $\mathcal{C}(G,\omega,K,\beta)$. Throughout this proof, we will fix the notation

$$A := A(K,\beta) \qquad \text{and} \qquad B := A(L,\psi)$$

for simplicity. Recall that $\Phi(f) = \mathrm{id}_A \otimes f \otimes \mathrm{id}_A$ for any morphism f in Vec_G^{ω}.

(a) Since the functor Φ is monoidal [Theorem 3.2], we have by Proposition 2.9(a) that $A^{K,\beta}(L,\psi) = \Phi(B)$ is an algebra in $\mathcal{C}(G,\omega,K,\beta)$, with multiplication and unit maps given by $\Phi(m_B)\Phi_{B,B}$ and $\Phi(u_B)\Phi_0$, respectively. Here, the monoidal structure of Φ is defined in Theorem 3.2, and in particular, the morphism $\Phi_{B,B}$ is given by means of the lift $\widetilde{\Phi}_{B,B}$ [Notation 3.1].

Note that

$$\begin{aligned}
&\widetilde{\Phi}_{B,B}[((\delta_f \otimes \delta_g) \otimes \delta_k) \otimes (\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'})] \\
&= (\alpha_{A,B,B} \otimes \mathrm{id}_A)\,(\mathrm{id}_{A,B} \otimes l_B \otimes \mathrm{id}_A)\,(\mathrm{id}_{A,B} \otimes \varepsilon_A m_A \otimes \mathrm{id}_{B,A}) \\
&\qquad (\mathrm{id}_{A,B} \otimes \alpha^{-1}_{A,A,B} \otimes \mathrm{id}_A)\,(\alpha_{AB,A,AB} \otimes \mathrm{id}_A)\,\alpha^{-1}_{ABA,AB,A} \\
&\qquad [((\delta_f \otimes \delta_g) \otimes \delta_k) \otimes ((\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'})] \\
&= \omega(fgk,f'g',k')\,\omega^{-1}(fg,k,f'g')\,\omega(k,f',g') \\
&\qquad (\alpha_{A,B,B} \otimes \mathrm{id}_A)\,(\mathrm{id}_{A,B} \otimes l_B \otimes \mathrm{id}_A)\,(\mathrm{id}_{A,B} \otimes \varepsilon_A m_A \otimes \mathrm{id}_{B,A}) \\
&\qquad [((\delta_f \otimes \delta_g) \otimes (\delta_k \otimes (\delta_{f'} \otimes \delta_{g'}))) \otimes \delta_{k'}] \\
&= \delta_{kf',e}\,\omega(fgk,f'g',k')\,\omega^{-1}(fg,k,f'g')\,\omega(k,f',g')\,\beta(k,f') \\
&\qquad (\alpha_{A,B,B} \otimes \mathrm{id}_A)(\mathrm{id}_{A,B} \otimes l_B \otimes \mathrm{id}_A) \\
&\qquad [((\delta_f \otimes \delta_g) \otimes (\delta_e \otimes \delta_{g'})) \otimes \delta_{k'}] \\
&= \delta_{kf',e}\,\omega(fgk,f'g',k')\,\omega^{-1}(fg,k,f'g')\,\omega(k,f',g')\,\beta(k,f')\,\omega^{-1}(f,g,g') \\
&\qquad (\delta_f \otimes (\delta_g \otimes \delta_{g'})) \otimes \delta_{k'}.
\end{aligned}$$

Therefore, the multiplication of $A^{K,\beta}(L,\psi)$ is given by

$$m_{A^{K,\beta}(L,\psi)}[((\delta_f \otimes \delta_g) \otimes \delta_k)\ \otimes_A\ ((\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'})]$$

$$= \delta_{kf',e}\, \omega(fgk, f'g', k')\, \omega^{-1}(fg, k, f'g')\, \omega(k, f', g')\, \omega^{-1}(f, g, g') \\ \beta(k, f')\, \psi(g, g')\, [(\delta_f \otimes \delta_{gg'}) \otimes \delta_{k'}].$$

Here, we use Proposition 5.7 for the multiplication and counit of A, and the monoidal structure of Vec_G^ω is given in Definition 5.2.

On the other hand, by using the definition of Φ_0 from Theorem 3.2 we get that the unit of $A^{K,\beta}(L,\psi)$ is given by

$$\begin{aligned} & u_{A^{K,\beta}(L,\psi)}(\delta_d) \;=\; \Phi(u_B)\, \Phi_0(\delta_d) \;=\; \Phi(u_B)\, (r_A^{-1} \otimes \mathrm{id}_A)\, \Delta_A(\delta_d) \\ & = \Phi(u_B)\, (r_A^{-1} \otimes \mathrm{id}_A) \bigoplus_{s\in K} \beta^{-1}(s^{-1}, s)\, \beta(d, s)\, \omega(s, s^{-1}, s)\, \omega(d, s, s^{-1})\, [\delta_{ds} \otimes \delta_{s^{-1}}] \\ & = \bigoplus_{s\in K} \beta^{-1}(s^{-1}, s)\, \beta(d, s)\, \omega(s, s^{-1}, s)\, \omega(d, s, s^{-1})\, [(\delta_{ds} \otimes \delta_e) \otimes \delta_{s^{-1}}] \\ & = \bigoplus_{s\in K} \beta^{-1}(s, s^{-1})\, \beta(d, s^{-1})\, \omega(s^{-1}, s, s^{-1})\, \omega(d, s^{-1}, s)\, [(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s] \\ & = \bigoplus_{s\in K} \beta^{-1}(s, s^{-1})\, \beta(d, s^{-1})\, \omega(ds, s, s^{-1})\, [(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s] \\ & = \bigoplus_{s\in K} \beta^{-1}(ds^{-1}, s)\, [(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s]. \end{aligned}$$

For the penultimate equation we used (5.4) with $(g_1, g_2, g_3, g_4) = (d, s^{-1}, s, s^{-1})$, and we used (5.6) with $\theta = \psi$ and $(f_1, f_2, f_3) = (ds^{-1}, s, s^{-1})$ for the last equation. Moreover, we use Proposition 5.7 for the comultiplication of A, and again the monoidal structure of Vec_G^ω is described in Definition 5.2.

(b) Since the functor Φ is Frobenius monoidal (see Theorem 3.2) and B is a Frobenius algebra in Vec_G^ω (see Proposition 5.7), $A^{K,\beta}(L,\psi) = \Phi(B)$ is a Frobenius algebra in $\mathcal{C}(G,\omega,K,\beta)$ by Proposition 2.9(c). Moreover, the comultiplication and counit of $A^{K,\beta}(L,\psi)$ determined by Φ are $\Phi^{B,B}\Phi(\Delta_B)$ and $\Phi^0\Phi(\varepsilon_B)$, respectively. Recall that the comonoidal structure $\Phi^{*,*}$ and Φ^0 of Φ is described in Theorem 3.2, and the structure of A and B are given in Proposition 5.7. Now,

$$\begin{aligned} & \Phi^{B,B}\ \Phi(\Delta_B)\, [(\delta_f \otimes \delta_g) \otimes \delta_k] \\ & = |L|^{-1} \bigoplus_{h\in L} \psi^{-1}(gh, h^{-1})\ \Phi^{B,B}\ [(\delta_f \otimes (\delta_{gh} \otimes \delta_{h^{-1}})) \otimes \delta_k]. \end{aligned} \tag{6.5}$$

Moreover, the lift $\widetilde{\Phi}^{B,B}$ of $\Phi^{B,B}$ on $(\delta_f \otimes (\delta_{gh} \otimes \delta_{h^{-1}})) \otimes \delta_k$ is given as follows:

$$\begin{aligned} & \widetilde{\Phi}^{B,B}[(\delta_f \otimes (\delta_{gh} \otimes \delta_{h^{-1}})) \otimes \delta_k] \\ & = \alpha_{ABA,AB,A}\, (\alpha^{-1}_{AB,A,AB} \otimes \mathrm{id}_A)\, (\mathrm{id}_{A,B} \otimes \alpha_{A,A,B} \otimes \mathrm{id}_A)\, (\mathrm{id}_{A,B} \otimes \Delta_A \otimes \mathrm{id}_{B,A}) \\ & \quad (\mathrm{id}_{A,B} \otimes u_A\, l_B^{-1} \otimes \mathrm{id}_A)\, (\alpha^{-1}_{A,B,B} \otimes \mathrm{id}_A)\, [(\delta_f \otimes (\delta_{gh} \otimes \delta_{h^{-1}})) \otimes \delta_k] \\ & = \alpha_{ABA,AB,A}\, (\alpha^{-1}_{AB,A,AB} \otimes \mathrm{id}_A)\, (\mathrm{id}_{A,B} \otimes \alpha_{A,A,B} \otimes \mathrm{id}_A)(\mathrm{id}_{A,B} \otimes \Delta_A \otimes \mathrm{id}_{B,A}) \\ & \quad \omega(f, gh, h^{-1})[((\delta_f \otimes \delta_{gh}) \otimes (\delta_e \otimes \delta_{h^{-1}})) \otimes \delta_k] \\ & = \bigoplus_{s\in K} \omega(f, gh, h^{-1})\, \omega(s, s^{-1}, s)\, \beta^{-1}(s^{-1}, s)\, \alpha_{ABA,AB,A}\, (\alpha^{-1}_{AB,A,AB} \otimes \mathrm{id}_A) \\ & \qquad (\mathrm{id}_{A,B} \otimes \alpha_{A,A,B} \otimes \mathrm{id}_A)[((\delta_f \otimes \delta_{gh}) \otimes ((\delta_s \otimes \delta_{s^{-1}}) \otimes \delta_{h^{-1}})) \otimes \delta_k] \\ & = \bigoplus_{s\in K} \omega(f, gh, h^{-1})\, \omega(s, s^{-1}, s)\, \omega^{-1}(s, s^{-1}, h^{-1})\, \omega(fgh, s, s^{-1}h^{-1}) \end{aligned}$$

$$\omega^{-1}(fghs, s^{-1}h^{-1}, k)\ \beta^{-1}(s^{-1}, s)\ [((\delta_f \otimes \delta_{gh}) \otimes \delta_s) \otimes ((\delta_{s^{-1}} \otimes \delta_{h^{-1}}) \otimes \delta_k)],$$
$$= \bigoplus_{s\in K} \omega(f, gh, h^{-1})\ \omega(fgh, s, s^{-1}h^{-1})\ \omega^{-1}(fghs, s^{-1}h^{-1}, k)$$
$$\omega^{-1}(s, s^{-1}, h^{-1})\ \beta^{-1}(s, s^{-1})\ [((\delta_f \otimes \delta_{gh}) \otimes \delta_s) \otimes ((\delta_{s^{-1}} \otimes \delta_{h^{-1}}) \otimes \delta_k)];$$

here, we used (5.6) with $\theta = \beta$ and $(f_1, f_2, f_3) = (s, s^{-1}, s)$ for the last equation. Together, with (6.5), we can normalize $\Phi^{B,B}\Phi_B(\Delta_B)$ by multiplying by $|K|^{-1}$ to get the desired formula for $\Delta_{A^{K,\beta}(L,\psi)}$.

On the other hand,

$$\begin{aligned}
&\varepsilon_{A^{K,\beta}(L,\psi)}[(\delta_f \otimes \delta_g) \otimes \delta_k] \\
&\quad = \Phi^0\Phi(\varepsilon_B)[(\delta_f \otimes \delta_g) \otimes \delta_k] \\
&\quad = \delta_{g,e}\ |L|\ m_A\ (r_A \otimes \mathrm{id}_A)[(\delta_f \otimes \delta_e) \otimes \delta_k] \\
&\quad = \delta_{g,e}\ |L|\ \beta(f, k)\ \delta_{fk}.
\end{aligned}$$
$\square$

Remark 6.6. Taking the forgetful functor $U : \mathcal{C}(G, \omega, \langle e\rangle, 1) \to \mathsf{Vec}_G^\omega$, observe that $U(A^{\langle e\rangle,1}(L, \psi)) \cong A(L, \psi)$ as algebras in Vec_G^ω.

The next result is expected, but we include it for the interest of the reader.

Proposition 6.7. *If ψ and ψ' are cohomologous 2-cocycles in $Z^2(L, \Bbbk^\times)$, then we get that $A^{K,\beta}(L, \psi) \cong A^{K,\beta}(L, \psi')$ as algebras in $\mathcal{C}(G, \omega, K, \beta)$.*

Proof. Let $\gamma : L \to \Bbbk^\times$ be a 1-cochain such that $d\gamma = \psi/\psi'$, that is,

$$\gamma(g_1)\gamma(g_2)\gamma^{-1}(g_1g_2) = \psi(g_1, g_2)\psi'^{-1}(g_1, g_2) \ \text{ for all } g_1, g_2 \in G. \tag{6.8}$$

Let $A := A(K, \beta),\ A_1 := A^{K,\beta}(L, \psi),\ A_2 := A^{K,\beta}(L, \psi')$ and let $\phi : A_1 \to A_2$ be defined by

$$\phi((\delta_f \otimes \delta_g) \otimes \delta_k) = \gamma(g)\ (\delta_f \otimes \delta_g) \otimes \delta_k,$$

which is clearly an A-bimodule map. We get that

$$\begin{aligned}
&m_{A_2}(\phi \otimes_A \phi)[((\delta_f \otimes \delta_g) \otimes \delta_k)\ \otimes_A\ ((\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'})] \\
&= m_{A_2}(\gamma(g)\gamma(g')\ [(\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'}))\ \otimes_A\ ((\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'}))]) \\
&= \gamma(g)\gamma(g')\delta_{kf',e}\ \omega(fgk, f'g', k')\ \omega^{-1}(fg, k, f'g')\ \omega(k, f', g')\ \omega^{-1}(f, g, g') \\
&\quad \cdot\ \beta(k, f')\ \psi'(g, g')\ [(\delta_f \otimes \delta_{gg'}) \otimes \delta_{k'}].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\phi(m_{A_1})[((\delta_f \otimes \delta_g) \otimes \delta_k)\ \otimes_A\ ((\delta_{f'} \otimes \delta_{g'}) \otimes \delta_{k'})] \\
&= \phi(\delta_{kf',e}\ \omega(fgk, f'g', k')\ \omega^{-1}(fg, k, f'g')\ \omega(k, f', g')\ \omega^{-1}(f, g, g') \\
&\quad \cdot\ \beta(k, f')\ \psi(g, g')\ [(\delta_f \otimes \delta_{gg'}) \otimes \delta_{k'}]) \\
&= \gamma(gg')\delta_{kf',e}\ \omega(fgk, f'g', k')\ \omega^{-1}(fg, k, f'g')\ \omega(k, f', g')\ \omega^{-1}(f, g, g') \\
&\quad \cdot\ \beta(k, f')\ \psi(g, g')\ [(\delta_f \otimes \delta_{gg'}) \otimes \delta_{k'}].
\end{aligned}$$

These are equal due to (6.8). Concerning the unit map in A_1, we have that

$$\phi(u_{A_1})(\delta_d)\ =\ \textstyle\bigoplus_{s\in K} \beta^{-1}(ds^{-1}, s)\ [(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s] = u_{A_2}(\delta_d).$$

since $\gamma(e) = 1$. So, ϕ is an algebra morphism. The inverse morphism $\phi^{-1} : A_2 \to A_1$ is given by

$$\phi^{-1}(\delta_f \otimes \delta_g \otimes \delta_k) = \gamma(g)^{-1}\ \delta_f \otimes \delta_g \otimes \delta_k.$$

Therefore, $A_1 \cong A_2$ as algebras in $\mathcal{C}(G, \omega, K, \beta)$. □

Next, we discuss algebraic properties of twisted Hecke algebras; see Section 2.2.

Proposition 6.9. *The twisted Hecke algebra $A^{K,\beta}(L, \psi)$, with structural morphisms $m, u, \Delta, \varepsilon$ given in Theorem 6.4, possesses the following properties:*

(a) *indecomposable;*
(b) *special; and*
(c) *separable.*

Proof. (a) By way of contradiction, suppose that $A^{K,\beta}(L, \psi) = A_1 \oplus A_2$ is a decomposable algebra. Then, A_1 contains as a summand a simple object $V_{g,\rho}$ from Lemma 6.2. In that result, we can take $f = k = e$ to get that $(\delta_e \otimes \delta_g) \otimes \delta_e$ is a summand of A_1 for some $g \in L$. Since A_1 is closed under multiplication, $m[((\delta_e \otimes \delta_g) \otimes \delta_e) \otimes_{A(L,\psi)} (\delta_e \otimes \delta_g) \otimes \delta_e)]$ is a summand of A_1. So, we get by Theorem 6.4(a), and by rescaling, that $(\delta_e \otimes \delta_{g^2}) \otimes \delta_e$ is a summand of A_1. Repeating this process, we obtain that $(\delta_e \otimes \delta_e) \otimes \delta_e$ is a summand of A_1 (as the element g has finite order in L). Likewise, A_2 contains as a summand a simple object $V_{g',\rho'}$ from Lemma 6.2, and we obtain that $(\delta_e \otimes \delta_{g'}) \otimes \delta_e$ is a summand of A_2 for some $g' \in L$ as a consequence. Arguing as above, $(\delta_e \otimes \delta_e) \otimes \delta_e$ is also a summand of A_2, which contradicts $A_1 \cap A_2 = (0)$. Therefore, $A^{K,\beta}(L, \psi)$ is an indecomposable algebra in $\mathcal{C}(G, \omega, K, \beta)$.

(b) To verify the special property, we compute:

$$\begin{aligned}
&m_{A^{K,\beta}(L,\psi)} \Delta_{A^{K,\beta}(L,\psi)}[(\delta_f \otimes \delta_g) \otimes \delta_k] \\
&= |K|^{-1}\, |L|^{-1} \bigoplus_{h \in L;\, s \in K} \omega(f, gh, h^{-1})\, \omega(fgh, s, s^{-1}h^{-1})\, \omega^{-1}(fghs, s^{-1}h^{-1}, k) \\
&\qquad\qquad \cdot\ \omega^{-1}(s, s^{-1}, h^{-1})\, \psi^{-1}(gh, h^{-1})\, \beta^{-1}(s, s^{-1}) \\
&\qquad\qquad \cdot\ m_{A^{K,\beta}(L,\psi)}[((\delta_f \otimes \delta_{gh}) \otimes \delta_s)\ \otimes_{A(K,\beta)}\ ((\delta_{s^{-1}} \otimes \delta_{h^{-1}}) \otimes \delta_k)] \\
&= |K|^{-1}\, |L|^{-1} \bigoplus_{h \in L;\, s \in K} \omega(f, gh, h^{-1})\, \omega(fgh, s, s^{-1}h^{-1})\, \omega^{-1}(fghs, s^{-1}h^{-1}, k) \\
&\qquad\qquad \cdot\ \omega^{-1}(s, s^{-1}, h^{-1})\, \psi^{-1}(gh, h^{-1})\, \beta^{-1}(s, s^{-1}) \\
&\qquad\qquad \cdot\ \omega(fghs, s^{-1}h^{-1}, k)\, \omega^{-1}(fgh, s, s^{-1}h^{-1})\, \omega(s, s^{-1}, h^{-1}) \\
&\qquad\qquad \cdot\ \omega^{-1}(f, gh, h^{-1})\, \psi(gh, h^{-1})\, \beta(s, s^{-1})\, ((\delta_f \otimes \delta_g) \otimes \delta_k) \\
&= |K|^{-1}\, |L|^{-1} \bigoplus_{h \in L;\, s \in K} (\delta_f \otimes \delta_g) \otimes \delta_k \\
&= (\delta_f \otimes \delta_g) \otimes \delta_k,
\end{aligned}$$

and for $A^{K,\beta}(L, \psi) = \bigoplus_{d \in K} \delta_d$, we get

$$\begin{aligned}
&\varepsilon_{A^{K,\beta}(L,\psi)}\, u_{A^{K,\beta}(L,\psi)}\, (\delta_d) \\
&= \varepsilon_{A^{K,\beta}(L,\psi)} \left(\textstyle\bigoplus_{s \in K} \beta^{-1}(ds^{-1}, s)[(\delta_{ds^{-1}} \otimes \delta_e) \otimes \delta_s] \right)
\end{aligned}$$

$$= \bigoplus_{s\in K} \beta^{-1}(ds^{-1}, s)\, |L|\, \beta(ds^{-1}, s)\, \delta_d$$
$$= |K|\, |L|\, \delta_d.$$

Therefore,

$$m_{A^{K,\beta}(L,\psi)}\, \Delta_{A^{K,\beta}(L,\psi)} = \mathrm{id}_{A^{K,\beta}(L,\psi)}, \quad \varepsilon_{A^{K,\beta}(L,\psi)}\, u_{A^{K,\beta}(L,\psi)} = |K|\, |L|\, \mathrm{id}_{A(K,\beta)}.$$

(c) This follows from Remark 2.11(b) and part (b) above. □

Now in comparison with Proposition 5.9(e), we ask:

Question 6.10. Under what conditions are the twisted Hecke algebras symmetric?

In order to address this question, see Definition 2.12 and consider Question 2.22. Now we examine the connected property of $A^{K,\beta}(L,\psi)$.

Proposition 6.11. *For the twisted Hecke algebra $A^{K,\beta}(L,\psi)$, it holds that*

$$\dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}(G,\omega,K,\beta)}(A(K,\beta), A^{K,\beta}(L,\psi))) = |K \cap L|.$$

As a consequence, $A^{K,\beta}(L,\psi)$ is connected precisely when $|K \cap L| = 1$.

Proof. Take $A := A(K,\beta)$, $B := A(L,\psi)$, and $\mathcal{C} := \mathsf{Vec}_G^{\omega}$. Recall that the free functor Φ from Theorem 3.2 is left adjoint to the forgetful functor $U : {}_A\mathcal{C}_A \to \mathcal{C}$ [Remark 3.3(c)]. So, $\mathrm{Hom}_{{}_A\mathcal{C}_A}(\Phi(B), A) \cong \mathrm{Hom}_{\mathcal{C}}(B, U(A))$. Since $\Phi(B) = A^{K,\beta}(L,\psi)$ and ${}_A\mathcal{C}_A = \mathcal{C}(G,\omega,K,\beta)$, we obtain that

$$\dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}(G,\omega,K,\beta)}(A, A^{K,\beta}(L,\psi))) = \dim_{\Bbbk}(\mathrm{Hom}_{\mathcal{C}}(B, A)) = |K \cap L|. \qquad \square$$

Recall that in the special case when $K = \langle e \rangle$, $\beta = 1$, the twisted Hecke algebra $A^{K,\beta}(L,\psi)$ is, via Remark 6.6, the twisted group algebra $A(L,\psi)$. Here, $\dim_{\Bbbk} \mathrm{Hom}_{{}_A\mathcal{C}_A}(A, A^{K,\beta}(L,\psi)) = 1$, so $A(L,\psi)$ is connected. This recovers Proposition 5.9(a).

7. Representation theory of group-theoretical fusion categories

We provide in this section a classification of indecomposable semisimple representations of group-theoretical fusion categories in terms of the twisted Hecke algebras defined and studied in Section 6; see Proposition 2.15 and Theorem 7.4 below. This result is analogous to Ostrik and Natale's classification of indecomposable semisimple representations of pointed fusion categories in terms of twisted group algebras (studied in Section 5) [29, 28]; see Theorem 7.3 below.

To begin, recall the notation from Sections 5 and 6, and consider the following notation.

Notation 7.1 (${}^x s$, ${}^x S$, ψ^x, Ω_x, $\mathcal{M}(L,\psi)$, $\mathcal{M}^{K,\beta}(L,\psi)$)**.**

- We write ${}^x s := xsx^{-1}$ and ${}^x S := \{{}^x s \colon s \in S\}$, for $x \in G$ and any set S.
- Take a 2-cochain ψ on a subgroup L of G and an element $x \in G$. The 2-cochain ψ^x on ${}^x L$ is defined by $\psi^x(h_1,h_2) = \psi({}^x h_1, {}^x h_2)$ for $h_1, h_2 \in L$.
- For $x \in G$, define the 2-cocycle $\Omega_x \colon G \times G \to \Bbbk^{\times}$ by

$$\Omega_x(h_1,h_2) = \frac{\omega({}^x h_1, {}^x h_2, x)\, \omega(x, h_1, h_2)}{\omega({}^x h_1, x, h_2)}.$$

- Let $\mathcal{M}(L,\psi)$ denote the left Vec_G^ω-module category consisting of right $A(L,\psi)$-modules in Vec_G^ω.
- Let $\mathcal{M}^{K,\beta}(L,\psi)$ denote the left $\mathcal{C}(G,\omega,K,\beta)$-module category consisting of right $A^{K,\beta}(L,\psi)$-modules in $\mathcal{C}(G,\omega,K,\beta)$.

Next, we borrow a condition from [28].

Definition 7.2 ($\mathcal{P}(G,\omega)$)**.** Let L, L' be subgroups of G. Take $\psi \in C^2(L,\Bbbk^\times)$ with $d\psi = \omega|_L$, and take $\psi' \in C^2(L',\Bbbk^\times)$ with $d\psi' = \omega|_{L'}$. We say that the pairs (L,ψ) and (L',ψ') are *conjugate* if there exists an element $x \in G$ so that

(a) $L = {}^xL'$, and
(b) the class of the 2-cocycle $\psi'^{-1}\ \psi^x\ \Omega_x|_{L'\times L'}$ is trivial in $H^2(L',\Bbbk^\times)$.

We denote by $\mathcal{P}(G,\omega)$ the set of conjugacy classes of pairs (L,ψ) as above.

Now consider the classification result for representations of pointed fusion categories mentioned above.

Theorem 7.3. [29, Example 2.1] [8, Example 9.7.2] [28]

(a) *We have that $\mathcal{M}(L,\psi)$ and $\mathcal{M}(L',\psi')$ are equivalent as Vec_G^ω-module categories if and only if $(L,\psi) = (L',\psi')$ in $\mathcal{P}(G,\omega)$.*
(b) *Every indecomposable, semisimple left module category over the pointed fusion category Vec_G^ω is equivalent to one of the form $\mathcal{M}(L,\psi)$, as left Vec_G^ω-module categories.* □

This brings us to the main result of this section, and of this article.

Theorem 7.4. *We have the following statements.*

(a) *$\mathcal{M}^{K,\beta}(L,\psi)$ and $\mathcal{M}^{K,\beta}(L',\psi')$ are equivalent as $\mathcal{C}(G,\omega,K,\beta)$-module categories if and only if $(L,\psi) = (L',\psi')$ in $\mathcal{P}(G,\omega)$.*
(b) *Every indecomposable, semisimple left module category over $\mathcal{C}(G,\omega,K,\beta)$ is equivalent to one of the form $\mathcal{M}^{K,\beta}(L,\psi)$, as left $\mathcal{C}(G,\omega,K,\beta)$-module categories.*

Proof. (a) By Theorem 7.3, we need to show that $\mathcal{M}(L,\psi)$ and $\mathcal{M}(L',\psi')$ are equivalent as Vec_G^ω-module categories if and only if $\mathcal{M}^{K,\beta}(L,\psi)$ and $\mathcal{M}^{K,\beta}(L',\psi')$ are equivalent as $\mathcal{C}(G,\omega,K,\beta)$-module categories. But this holds by using Theorem 4.9, with Propositions 5.7 and 5.9(c), applied to $\mathcal{C} = \mathsf{Vec}_G^\omega$, $A = A(K,\beta)$, $B = A(L,\psi)$, and $B' = A(L',\psi')$.

(b) For a fusion category $\mathcal{D}$, let $\mathsf{Indec}(\mathsf{Mod}(\mathcal{D}))$ denote a set of equivalence class representatives of indecomposable semisimple left $\mathcal{D}$-module categories, and let $[\mathcal{M}]$ be the class of $\mathcal{D}$-module categories equivalent to $\mathcal{M}$ (as left $\mathcal{D}$-module categories).

Now by Theorem 7.3 and [26, Sections 3 and 4] (see also [8, Theorem 7.12.11]), there is a 1-to-1 correspondence between the finite sets,

$$\mathsf{Indec}(\mathsf{Mod}(\mathcal{C}(G,\omega,K,\beta))) \qquad \text{and} \qquad \mathcal{P}(G,\omega);$$

namely, both of these sets are in bijection with $\mathsf{Indec}(\mathsf{Mod}(\mathsf{Vec}_G^\omega))$. On the other hand, since $A^{K,\beta}(L,\psi)$ is an indecomposable and separable algebra in $\mathcal{C}(G,\omega,K,\beta)$ [Proposition 6.9], the finite collection

$$\{[\mathcal{M}^{K,\beta}(L,\psi)]\}_{(L,\psi)\in\mathcal{P}(G,\omega)}$$

consists of equivalence classes of indecomposable semisimple left $\mathcal{C}(G,\omega,K,\beta)$-module categories [Proposition 2.15]. (Indeed, indecomposability and semisimplicity are preserved under module category equivalence.) Moreover, by (a), this collection is also in bijection with the finite set $\mathcal{P}(G,\omega)$. Therefore, as finite sets,

$$\mathsf{Indec}(\mathsf{Mod}(\mathcal{C}(G,\omega,K,\beta))) = \{[\mathcal{M}^{K,\beta}(L,\psi)]\}_{(L,\psi)\in\mathcal{P}(G,\omega)},$$

and this verifies part (b). $\square$

To compare the twisted Hecke algebras in $\mathcal{C}(G,\omega,K,\beta)$ with the algebras in Proposition 2.26, we have the following remark.

Remark 7.5. If A is an algebra in a rigid monoidal category $\mathcal{C}$ such that $A = A^* = {}^*A$ and B is a coalgebra in $\mathcal{C}$, then we can give a coalgebra structure in $({}^*A \otimes B) \otimes A$ such that this is Frobenius when B is Frobenius. Also, if A and B are Frobenius in $\mathcal{C}$ with A special, then $({}^*A \otimes B) \otimes A$ is isomorphic to $(A \otimes B) \otimes A$ in ${}_A\mathcal{C}_A$ and therefore is Frobenius. Now by Propositions 2.26, 5.7, and 5.9(c), and Theorem 7.4, the (Frobenius) algebras $({}^*A(K,\beta) \otimes A(L,\psi)) \otimes A(K,\beta)$ serve as Morita equivalence class representatives of indecomposable, separable algebras in $\mathcal{C}(G,\omega,K,\beta)$.

Finally, we compare our work with recent work of P. Etingof, R. Kinser, and the last author in [9].

Remark 7.6. Morita equivalence class representatives of indecomposable, separable algebras in group-theoretical fusion categories $\mathcal{C}$ were used in the recent study of tensor algebras in $\mathcal{C}$; see [9, Theorem 3.11 and Section 5]. (Note that a 'separable algebra' here is the same as a 'semisimple algebra' in [9] as we are working over an algebraically closed field.) Now by Theorem 7.4, our construction of the twisted Hecke algebras in $\mathcal{C}$ serve as the base algebras of tensor algebras in $\mathcal{C}$, up to the notion of equivalence given in [9, Definition 3.4].

Example 7.7. Continuing the remark above, let $\mathsf{Rep}(H_8)$ be the category of finite-dimensional representations of the Kac-Paljutkin Hopf algebra, which is a group-theoretical fusion category $\mathcal{C}(D_8,\omega,\mathbb{Z}_2,1)$; see [9, Example 5.3 and Section 5.3] for more details. A collection of Morita equivalence class representatives of indecomposable, separable algebras (or, up to equivalence, of base algebras of the tensor algebras) in $\mathsf{Rep}(H_8)$ is given in [9, Theorem 5.23]. The correspondence of those six algebras with the conjugacy classes of pairs (L,ψ) is presented in [9, Proposition 5.26]. Thus, we can replace the algebras in [9, Theorem 5.23] corresponding to such pairs (L,ψ) with the twisted Hecke algebras $A^{\mathbb{Z}_2,1}(L,\psi)$ featured here. The advantage is that the six algebras of [9, Theorem 5.23] were found via ad-hoc methods [9, Remark 5.28], whereas our construction provides a uniform collection of Morita equivalence classes representatives of algebras in $\mathsf{Rep}(H_8)$.

Appendix A. Remainder of the proof of Theorem 4.1

In this appendix, we fill in some details for the proof of Theorem 4.1.

Proposition A.1. *We have that*

$$(P, \lambda_P^{\Gamma(S)},\ \rho_P^{\Gamma(S')}) \in {}_{\Gamma(S)}\mathcal{T}_{\Gamma(S')}, \qquad (Q, \lambda_Q^{\Gamma(S')},\ \rho_Q^{\Gamma(S)}) \in {}_{\Gamma(S')}\mathcal{T}_{\Gamma(S)}, \qquad \textit{where}$$

$$\begin{aligned} \lambda_P^{\Gamma(S)} &= \Gamma(\lambda^{S}_{\overline{P}})\, \Gamma_{S,\overline{P}} : \Gamma(S) \otimes_{\mathcal{T}} P \to P, & \rho_P^{\Gamma(S')} &= \Gamma(\rho^{S'}_{\overline{P}})\, \Gamma_{\overline{P},S'} : P \otimes_{\mathcal{T}} \Gamma(S') \to P, \\ \lambda_Q^{\Gamma(S')} &= \Gamma(\lambda^{S'}_{\overline{Q}})\, \Gamma_{S',\overline{Q}} : \Gamma(S') \otimes_{\mathcal{T}} Q \to Q, & \rho_Q^{\Gamma(S)} &= \Gamma(\rho^{S}_{\overline{Q}})\, \Gamma_{\overline{Q},S} : Q \otimes_{\mathcal{T}} \Gamma(S) \to Q. \end{aligned}$$

Proof. It is straight-forward to check that P is a right $\Gamma(S')$-module in $\mathcal{T}$ with action given by $\rho_P^{\Gamma(S')}$. In a similar way, it can be seen that P is a left $\Gamma(S)$-module in $\mathcal{T}$ with action $\lambda_P^{\Gamma(S)}$. Let us now check the left and right action compatibility for P. Consider the diagram, where $\otimes := \otimes_{\mathcal{S}}$ and we suppress the $\otimes_*$ symbols in morphisms below.

$$\begin{tikzcd}
(\Gamma(S) \otimes_{\mathcal{T}} P) \otimes_{\mathcal{T}} \Gamma(S') \arrow[rrrr, "\alpha_{\Gamma(S),P,\Gamma(S')}"] \arrow[dr, "\Gamma_{S,\overline{P}}\mathrm{id}"] \arrow[ddd, "\lambda_P^{\Gamma(S)}\mathrm{id}"'] & & & & \Gamma(S) \otimes_{\mathcal{T}} (P \otimes_{\mathcal{T}} \Gamma(S')) \arrow[dl, "\mathrm{id}\,\Gamma_{\overline{P},S'}"'] \arrow[ddd, "\mathrm{id}\,\rho_P^{\Gamma(S')}"] \\
& \Gamma(S \otimes \overline{P}) \otimes_{\mathcal{T}} \Gamma(S') \arrow[d, "\Gamma_{S\otimes\overline{P},S'}"] \arrow[ddl, "\Gamma(\lambda^{S}_{\overline{P}})\mathrm{id}"'] \arrow[rr, phantom, "(1)"] & & \Gamma(S) \otimes_{\mathcal{T}} \Gamma(\overline{P} \otimes S') \arrow[d, "\Gamma_{S,\overline{P}\otimes S}"'] \arrow[ddr, "\mathrm{id}\,\Gamma(\rho^{S'}_{\overline{P}})"] & \\
& \Gamma((S \otimes \overline{P}) \otimes S') \arrow[rr, "\Gamma(\alpha_{S,\overline{P},S'})"'] \arrow[d, "\Gamma(\lambda^{S}_{\overline{P}}\ \mathrm{id})"] \arrow[dl, phantom, "(3)"] & & \Gamma(S \otimes (\overline{P} \otimes S')) \arrow[d, "\Gamma(\mathrm{id}\ \rho^{S'}_{\overline{P}})"'] \arrow[dr, phantom, "(4)"] & \\
P \otimes_{\mathcal{T}} \Gamma(S') \arrow[r, "\Gamma_{\overline{P},S'}"] \arrow[drr, "\rho_P^{\Gamma(S')}"'] & \Gamma(\overline{P} \otimes S') \arrow[dr, "\Gamma(\rho^{S'}_{\overline{P}})"] \arrow[rr, phantom, "(2)"] & & \Gamma(S \otimes \overline{P}) \arrow[dl, "\Gamma(\lambda^{S}_{\overline{P}})"'] & \Gamma(S) \otimes_{\mathcal{T}} P \arrow[l, "\Gamma_{S,\overline{P}}"'] \arrow[dll, "\lambda_P^{\Gamma(S)}"] \\
& & P & &
\end{tikzcd}$$

Here, (1) commutes as Γ is a monoidal functor, and (2) commutes since $\overline{P} \in {}_{S}\mathcal{C}_{S'}$. The diagrams (3) and (4) commute due to the naturality of $\Gamma_{*,*}$, and the triangles correspond to the definition of the left and right actions of P in ${}_{\Gamma(S)}\mathcal{T}_{\Gamma(S')}$. Therefore, $(P, \lambda_P^{\Gamma(S)},\ \rho_P^{\Gamma(S')}) \in {}_{\Gamma(S)}\mathcal{T}_{\Gamma(S')}$. Analogously, $(Q, \lambda_Q^{\Gamma(S')},\ \rho_Q^{\Gamma(S)}) \in {}_{\Gamma(S')}\mathcal{T}_{\Gamma(S)}$. $\square$

Proposition A.2. *The epimorphisms*

$$\begin{aligned} \tau : P \otimes_{\Gamma(S')} Q &\twoheadrightarrow \Gamma(S) \quad \in {}_{\Gamma(S)}\mathcal{T}_{\Gamma(S)}, \\ \mu : Q \otimes_{\Gamma(S)} P &\twoheadrightarrow \Gamma(S') \quad \in {}_{\Gamma(S')}\mathcal{T}_{\Gamma(S')}, \end{aligned}$$

satisfy diagrams $(*)$ *and* $(**)$ *in Proposition 2.25(b).*

Proof. Diagram $(*)$ corresponds to the following; $\otimes$ is understood from context:

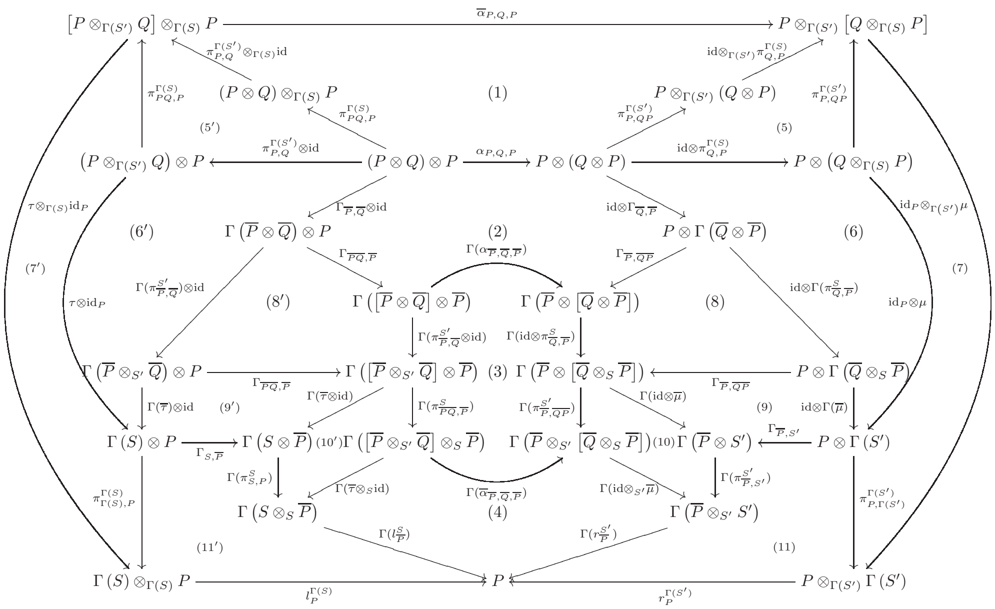

Diagram (1) is the definition of $\overline{\alpha}$ (see Definition 2.23). Diagram (2) commutes as Γ is a monoidal functor, and (3) results from applying Γ to the definition of $\overline{\alpha}$. Diagram (4) is the result of applying the functor Γ to the diagram (*). Diagrams (5) and (7) follow from (2.21). Diagram (6) is (4.2). Diagrams (8) and (9) commute from naturality of $\Gamma_{*,*}$. Diagram (10) commutes by applying Γ to (2.21). The proof of diagram (11) is given below. Finally, the commutativity of (5′)–(11′) follow analogously to the proof of (5)–(11), respectively. Therefore, diagram $(*)$ commutes. In an analogous manner, diagram $(**)$ commutes.

$$\begin{tikzcd}
\Gamma(\overline{P}) \otimes_{\mathcal{T}} \Gamma(S') \arrow[rrrr, "\Gamma_{\overline{P},S'}"] \arrow[d, "\pi^{\Gamma(S')}_{\Gamma(\overline{P}),\Gamma(S')}"'] \arrow[ddrr, bend left, "\rho^{\Gamma(S')}_{\Gamma(\overline{P})}"] & & & & \Gamma(\overline{P} \otimes_{\mathcal{S}} S') \arrow[d, "\Gamma(\pi^{S'}_{\overline{P},S'})"] \arrow[ddll, bend right, "\Gamma(\rho^{S'}_{\overline{P}})"'] \\
\Gamma(\overline{P}) \otimes_{\Gamma(S')} \Gamma(S') \arrow[drr, "r^{\Gamma(S')}_{\Gamma(\overline{P})}"'] & \text{(def. of } r^{\Gamma(S')}_{\Gamma(\overline{P})}) & \text{(def. of } \rho_P^{\Gamma(S')}) & \text{(def. of } r^{S'}_{\overline{P}}) & \Gamma(\overline{P} \otimes_{S'} S') \arrow[dll, "\Gamma(r^{S'}_{\overline{P}})"] \\
& & \Gamma(\overline{P}) & &
\end{tikzcd}$$

□

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